Homework 9

Exercise 1

Let $\lambda \in \mathbb{F}$ (the field) and $X \in V$ such that $PX = \lambda X$. Then $\lambda X = PX = P^2 X = P(PX) = P(\lambda X)$ $= \lambda PX = \lambda^2 X$.

Thus, $\lambda = \lambda^2$, which means $\lambda = 0$ or $\lambda = 1$.

Exercise 2

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ and suppose that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} x\cos\theta + y\sin\theta = x \\ x\sin\theta - y\cos\theta = y \end{cases}$$

$$\Leftrightarrow \begin{cases} x(1-\cos\theta) = y\sin\theta \\ y(1+\cos\theta) = x\sin\theta \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y\dfrac{\sin\theta}{1-\cos\theta} \quad \text{if } \cos\theta \neq 1 \\ x = y\dfrac{1+\cos\theta}{\sin\theta} = y\dfrac{(1+\cos\theta)(1-\cos\theta)}{\sin\theta(1-\cos\theta)} = y\dfrac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)} = y\dfrac{\sin\theta}{1-\cos\theta}. \end{cases}$$

Thus, a possible eigenvector is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sin\theta}{1-\cos\theta} \\ 1 \end{pmatrix}$.

If $\cos\theta = 1$, then the initial system reduces to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, and has a solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Thus, for any θ, the above matrix admits the eigenvalue 1.

Exercise 3

For $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ one has $L_A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} 2x \\ 3x + 4y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y = -\frac{3}{2} x \end{cases}$

All eigenvectors are given by $\begin{pmatrix} x \\ -\frac{3}{2} x \end{pmatrix}$, $x \in \mathbb{R} \setminus \{0\}$.

Similarly $\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 4 \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x = 0 \\ y \text{ arbitrary} \end{cases}$

All eigenvectors are given by $\begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in \mathbb{R} \setminus \{0\}$.

For $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, one has $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y = 0 \\ z = 0 \end{cases}$

The corresponding eigenspace is $\underline{\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x \in \mathbb{R} \right\}}$.

For the eigenvalue 5, one obtains the system

$\begin{cases} 4x = y \\ y \text{ arbitrary} \\ z = 0 \end{cases}$ and the corresponding eigenspace

is $\underline{\left\{ \begin{pmatrix} y/4 \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \mathbb{R} \right\}}$.

For the eigenvalue 7 one finds the eigenspace

$\underline{\left\{ \begin{pmatrix} -\frac{1}{2} y \\ y \\ -2y \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \mathbb{R} \right\}}$.

Exercise 4

1) Yes, with eigenvalue λ^3.
2) Yes, since $(A + 2\mathbb{1})X = (\lambda + 2)X$, the corresponding eigenvalue is $\lambda + 2$.
3) Yes, with eigenvalue 4λ.
4) No, if 0 is an eigenvalue, then $AX = 0$ has a solution $X \neq 0$, and thus $\ker(L_A) \neq \{0\}$
$\Rightarrow A$ can not be invertible.
5) Since $\mathbb{1}_n X = A^{-1} A X = A^{-1} \lambda X$
$\Rightarrow A^{-1} X = \frac{1}{\lambda} X$, then X is an eigenvector of L_A with the eigenvalue λ^{-1}.
6) $\ker(L_A - \lambda \mathbb{1}) \neq \{0\}$ since $X \in \mathrm{Ker}(L_A - \lambda\mathbb{1})$.
7) $\mathrm{Det}(A - \lambda \mathbb{1}_n) = 0$ since, by the previous question, $(A - \lambda \mathbb{1}_n)$ is not invertible.

Exercise 5

No, it is impossible to have $A, B \in M_n(\mathbb{R})$ with $AB - BA = \mathbb{1}_n$. Indeed, by taking the trace on both sides of this equality one would get
$\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0$ but $\mathrm{Tr}(\mathbb{1}_n) = n$, which means $0 = n$!