# Homework 9

**Exercise 1** *Let* $\mathrm{P}: V \to V$ *be a linear map on a vector space* $V$ *and assume that* $\mathrm{P}$ *is a projection. Show that* $\mathrm{P}$ *can only have two possible eigenvalues, namely* $0$ *and* $1$.

**Exercise 2** *For any* $\theta \in [0, 2\pi)$, *consider the matrix* $\mathcal{A}(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ *and show that the corresponding linear map* $\mathrm{L}_{\mathcal{A}(\theta)} : \mathbb{R}^2 \to \mathbb{R}^2$ *always admits the eigenvalue* $1$.

**Exercise 3** *Consider the matrix* $\mathcal{A} := \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$ *and show that* $2$ *and* $4$ *are eigenvalues of the associated linear map* $\mathrm{L}_{\mathcal{A}}$. *What are all corresponding eigenvectors ? Similarly, consider the matrix* $\mathcal{B} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$ *and show that* $1$, $5$ *and* $7$ *are eigenvalues of the associated linear map. Determine the corresponding eigenspaces.*

**Exercise 4** *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *be invertible, and assume that* $\lambda \in \mathbb{R}$ *is an eigenvalue of* $\mathrm{L}_{\mathcal{A}}$ *with* $X \in \mathbb{R}^n$ *a corresponding eigenvector.*

1. *Is* $X$ *an eigenvector of* $\mathrm{L}_{\mathcal{A}^3}$ *? If so, what is the corresponding eigenvalue ?*
2. *Is* $X$ *an eigenvector of the linear map associated with* $\mathcal{A} + 2\mathbf{1}_n$ *? If so, what is the corresponding eigenvalue ?*
3. *Is* $X$ *an eigenvector of* $\mathrm{L}_{4\mathcal{A}}$ *? If so, what is the corresponding eigenvalue ?*
4. *Can* $\lambda$ *be equal to* $0$ *?*
5. *Is* $X$ *an eigenvector of* $\mathrm{L}_{\mathcal{A}^{-1}}$ *? If so, what is the corresponding eigenvalue ?*
6. *What can you say about* $\mathsf{Ker}(\mathrm{L}_{\mathcal{A}} - \lambda\mathbf{1})$ *?*
7. *What can you say about* $\mathsf{Det}(\mathcal{A} - \lambda\mathbf{1}_n)$ *?*

**Exercise 5** *Do there exist* $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{R})$ *such that* $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \mathbf{1}_n$ *? Justify your answer.*