Homework 8

Exercise 1

Let us first observe that $\mathcal{A}X = B \iff$

$$\left(\mathcal{A}^1 \mathcal{A}^2 \dots \mathcal{A}^n\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \mathcal{A}^1 + x_2 \mathcal{A}^2 + \dots + x_n \mathcal{A}^n = B.$$

Thus, by inserting this equality in the next computation and by using the properties on the determinant one gets:

$$\mathrm{Det}(\mathcal{A}^1 \dots \underset{\substack{\uparrow \\ j^{th}\text{ position}}}{B} \dots \mathcal{A}^n) = \mathrm{Det}(\mathcal{A}^1 \dots \underbrace{x_1\mathcal{A}^1 + x_2\mathcal{A}^2 + \dots x_n\mathcal{A}^n}_{j^{th}\text{ position}} \dots \mathcal{A}^n)$$

$$= \mathrm{Det}(\mathcal{A}^1 \dots x_j\mathcal{A}^j \dots \mathcal{A}^n) \quad \text{because all the other terms are } 0$$

$$= x_j\,\mathrm{Det}(\mathcal{A}^1 \dots \mathcal{A}^j \dots \mathcal{A}^n)$$

$$= x_j\,\mathrm{Det}(\mathcal{A}).$$

Since $\mathrm{Det}(\mathcal{A}) \neq 0$, one concludes that

$$x_j = \frac{\mathrm{Det}(\mathcal{A}^1 \dots B \dots \mathcal{A}^n)}{\mathrm{Det}(\mathcal{A})}.$$

Exercise 2

If $\mathcal{A}$ is invertible, one has

$$1 = \mathrm{Det}(1_n) = \mathrm{Det}(\mathcal{A}\mathcal{A}^{-1}) = \mathrm{Det}(\mathcal{A})\,\mathrm{Det}(\mathcal{A}^{-1})$$

from which one deduces that

$$\mathrm{Det}(\mathcal{A}^{-1}) = \frac{1}{\mathrm{Det}(\mathcal{A})}.$$

Exercise 3

Let $A, A', B \in M_n(\mathbb{F})$ with B invertible, and assume that $A' = BAB^{-1}$. Then one has

$$\underline{\mathrm{Det}(A')} = \mathrm{Det}(BAB^{-1}) = \mathrm{Det}(B)\,\mathrm{Det}(A)\,\mathrm{Det}(B^{-1})$$

$$= \underbrace{\mathrm{Det}(B)\,\mathrm{Det}(B^{-1})}_{=1 \text{ by Exercise 2}}\,\mathrm{Det}(A) = \underline{\mathrm{Det}(A)}.$$

they are just numbers so they commute.

Exercise 4 One has

$$\mathrm{Det}\begin{pmatrix} x+1 & x-1 \\ x & 2x+5 \end{pmatrix} = (x+1)(2x+5) - x(x-1)$$

$$= 2x^2+5x+2x+5-x^2+x = \underline{x^2+8x+5}$$

Exercise 6 From the equality

$$(V_1'\,V_2'\,\dots\,V_n') = (BV_1\;BV_2\,\dots\,BV_n)$$
$$= B(V_1\,V_2\,\dots\,V_n)$$

and since $\mathrm{Det}(BA) = \mathrm{Det}(B)\,\mathrm{Det}(A)$ for any $A, B \in M_n(\mathbb{R})$ one deduces from Exercise 5 that

$$\mathrm{Vol}(V_1'\,V_2'\dots V_n') = |\mathrm{Det}(V_1'\,V_2'\dots V_n')| = |\mathrm{Det}(B(V_1\,V_2\dots V_n))|$$

$$= |\mathrm{Det}(B)\,\mathrm{Det}(V_1\,V_2\dots V_n)|$$

$$= |\mathrm{Det}(B)|\;\mathrm{Vol}(V_1\,V_2\dots V_n).$$

Thus, $|\mathrm{Det}(B)|$ is related to the change of the volume of the n-dimensional boxes generated either by $V_1, V_2, \dots, V_n$ or by $V_1', V_2', \dots, V_n'$.

Exercise 5

Consider $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $\tilde{X} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$, $\tilde{Y} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$.

Then $\tilde{X} \times \tilde{Y} = \begin{pmatrix} 0 \\ 0 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$, and

$\| \tilde{X} \times \tilde{Y} \| = | x_1 y_2 - x_2 y_1 |$ = area of the parallelogram spanned by $\tilde{X}$ and $\tilde{Y}$, i.e. area of the parallelogram spanned by X and Y.

On the other hand, $\mathrm{Det} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$, and therefore

$$\underline{\mathrm{Vol}(X, Y) = |\det(X, Y)|}.$$

For the second question, observe that from some heuristic considerations, one should have:

- $\mathrm{Vol}(\lambda X_1, X_2, \dots, X_n) = \lambda \, \mathrm{Vol}(X_1, X_2, \dots, X_n)$
  (rescalling of one vector)
- $\mathrm{Vol}(X_1 + X_1', X_2, \dots, X_n) = \mathrm{Vol}(X_1, \dots, X_n) + \mathrm{Vol}(X_1', \dots, X_n)$
  (addition of volume)

  and these 2 properties hold for any variable.
- $\mathrm{Vol}(X_1, X_1, X_3, \dots, X_n) = 0$ since the n-dimensional box is then flat, and this hold whenever two vectors X_j and X_k are equal.
- $\mathrm{Vol}(E_1, E_2, \dots, E_n) = 1$ (normalization).

Since the determinant is the only map having these properties and since the volume should be positive, one deduces the statement.