Nagoya University, G30 program Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 8

Exercise 1 *Prove Cramer's rule, i.e. show that if $\mathcal{A} \in M_n(\mathbb{F})$ is invertible and if $X \in \mathbb{F}^n$ satisfies $\mathcal{A}X = B$ for some $B \in \mathbb{F}^n$, then*

$$x_j = \frac{1}{\mathsf{Det}(\mathcal{A})} \mathsf{Det}\big(\mathcal{A}^1\, \mathcal{A}^2 \dots B \dots \mathcal{A}^n\big),$$

where B is replacing the column $\mathcal{A}^j$. For that purpose, one should first recall that $\mathcal{A}X = B$ is equivalent to $x_1\mathcal{A}^1 + x_2\mathcal{A}^2 + \dots + x_n\mathcal{A}^n = B$, and insert this equality in the term $\mathsf{Det}\big(\mathcal{A}^1\, \mathcal{A}^2 \dots B \dots \mathcal{A}^n\big)$.

Exercise 2 *Show that if $\mathcal{A} \in M_n(\mathbb{F})$ is invertible then the following equality holds:*

$$\mathsf{Det}\left(\mathcal{A}^{-1}\right) = \frac{1}{\mathsf{Det}(\mathcal{A})}.$$

Exercise 3 *Show that two similar square matrices share the same determinant.*

Exercise 4 *Compute the determinant of the matrix $\left(\begin{smallmatrix} x+1 & x-1 \\ x & 2x+5 \end{smallmatrix}\right)$.*

Exercise 5 *Let X, Y be two vectors in $\mathbb{R}^2$. Check that the area of the parallelogram spanned by X and Y is equal to the absolute value of the determinant of the matrix $(X\ Y) \in M_2(\mathbb{R})$. More generally, if $X_1, \dots, X_n$ are n vectors of $\mathbb{R}^n$, one writes $\mathsf{Vol}(X_1, \dots, X_n)$ for the volume of the n-dimensional box spanned by $X_1, \dots, X_n$. Why is it natural to have*

$$\mathsf{Vol}(X_1, \dots, X_n) = |\,\mathsf{Det}(X_1 \dots X_n)|\ ?$$

Exercise 6 *Let $\{V_1, \dots, V_n\}$ and $\{V_1', \dots, V_n'\}$ be two bases of $\mathbb{R}^n$, and let $\mathcal{B} \in M_n(\mathbb{R})$ be the matrix of change of bases, i.e. $V_j' = \mathcal{B}V_j$ for any $j = 1, 2, \dots, n$. What is the geometric interpretation of $|\,\mathsf{Det}(\mathcal{B})|$ in this setting ? For that purpose, one should first check that if $(V_1\ V_2 \dots V_n)$ denotes the matrix with columns V_j and $(V_1'\ V_2' \dots V_n')$ denotes the matrix with columns V_j', then one has*

$$(V_1'\ V_2' \dots V_n') = (\mathcal{B}V_1\ \mathcal{B}V_2 \dots \mathcal{B}V_n) = \mathcal{B}(V_1\ V_2 \dots V_n).$$