Homework 7

Exercise 1

Clearly, by using iteratively the first formula one gets

$$\mathrm{Det}(\mathbb{1}_n) = 1 \, \mathrm{Det}(\mathbb{1}_n(1,1)) + 0 \, \mathrm{Det}(\mathbb{1}_n(2,1)) + 0 \ldots + 0$$

$$= 1 \, \mathrm{Det}(\mathbb{1}_{n-1}) = 1 \cdot 1 \cdot \mathrm{Det}(\mathbb{1}_{n-2})$$

$$= \ldots = 1 \cdot 1 \cdot 1 \ldots \cdot \underbrace{\mathrm{Det}(\mathbb{1}_1)}_{=1} = 1.$$

i) For the linearity, consider $A = (A^1 \ldots \underbrace{A^j + B^j}_{\text{colum } j} \ldots A^n)$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1j}+b_{1j} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{n1} & & a_{nj}+b_{nj} & \cdots & a_{nn} \end{pmatrix}$$

and expand with respect to the column j:

$$\mathrm{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} (a_{ij} + b_{ij}) \, \mathrm{Det}(A(i,j))$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \, \mathrm{Det}(A(i,j)) + \sum_{i=1}^{n} (-1)^{i+j} b_{ij} \, \mathrm{Det}(A(i,j))$$

$$= \mathrm{Det}(A^1 \ldots A^j \ldots A^n) + \mathrm{Det}(A^1 \ldots B^j \ldots A^n)$$

Similarly $\mathrm{Det}(A^1 \ldots \lambda A^j \ldots A^n)$

$$= \sum_{i=1}^{n} (-1)^{i+j} \lambda a_{ij} \, \mathrm{Det}(A(i,j))$$

$$= \lambda \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \, \mathrm{Det}(A(i,j)) = \lambda \, \mathrm{Det}(A^1 \ldots A^n).$$

The determinant is thus linear in the column j.

Since j is arbitrary, it follows that the determinant is linear in any column of $\mathcal{A}$.

ii) The proof is performed by induction.

If $n = 2$, one has $\mathrm{Det}\begin{pmatrix} a & a \\ b & b \end{pmatrix} \overset{\uparrow}{=} ab - ba = 0$.

by the formula

Assume now that $\mathrm{Det}(\mathcal{A}) = 0$ for any $\mathcal{A} \in M_{n-1}(\mathbb{F})$ whenever two columns of $\mathcal{A}$ are equal, and let us prove that the same conclusion holds for any $\mathcal{A} \in M_n(\mathbb{F})$ whenever $\mathcal{A}$ has two columns equal.

For siplicity, assume that $\mathcal{A}^l = \mathcal{A}^k$ (column l = column k) and let us choose $j \neq l$ and $j \neq k$. By expanding $\mathrm{Det}(\mathcal{A})$ with respect to the column j, one has

$$\mathrm{Det}(\mathcal{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \,\mathrm{Det}(\mathcal{A}(i,j))$$

But $\mathcal{A}(i,j)$ has still the columns l and k equal, and thus by assumption $\mathrm{Det}(\mathcal{A}(i,j)) = 0$.

<u>Conclusion</u>: $\mathrm{Det}(\mathcal{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\, 0 = 0$.

<u>Exercice 2</u> $\mathrm{Det}(\mathcal{A}) = \mathrm{Det}\begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & a_{22} & a_{33} & \vdots \\ 0 & & \ddots & a_{nn} \end{pmatrix} =$

$$= a_{11}\,\mathrm{Det}\begin{pmatrix} a_{22} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix} = a_{11} a_{22}\,\mathrm{Det}\begin{pmatrix} a_{33} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}$$

$$= a_{11} a_{22} a_{33} \cdots a_{nn}.$$

Thus $\mathrm{Det}(\mathcal{A})$ = product of the elements on its diagonal.

Exercise 3

1) $\mathrm{Det}\begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots\, c & & \\ & & & 1 & \\ 0 & & & & \ddots\, 1 \end{pmatrix} \underset{\uparrow}{=} c \,\mathrm{Det}(\mathbb{1}_n) = c$

by multilinearity

2) $\mathrm{Det}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & \cdots & 1 & \\ & & \vdots & & \vdots & \\ & & 1 & \cdots & 0 & \\ & & & & & 1 \ddots 1 \end{pmatrix} \underset{\uparrow}{=} -\mathrm{Det}(\mathbb{1}_n) = -1$

because 2 rows have been exchanged, starting from the identity matrix.

3) $\mathrm{Det}\begin{pmatrix} 1 & & & 0 \\ & 1 \ddots & & \\ & \vdots & & \\ 0 & c \cdots & 1 & \\ & & & \ddots 1 \end{pmatrix} \underset{\uparrow}{=} \mathrm{Det}(\mathbb{1}_n) = 1$

because one has added c times the column r to the column s. Adding a linear combination of other columns to one column does not change the determinant.

Exercise 4

a) $\mathrm{Det}\begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 5 & 7 \end{pmatrix} = -2\,\mathrm{Det}\begin{pmatrix} -1 & 1 \\ 5 & 7 \end{pmatrix} = -2(-12) = \underline{24}$,

b) $\mathrm{Det}\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 8 \end{pmatrix} = \underline{0}$ cf ex 3,

c) $\mathrm{Det}\begin{pmatrix} 3 & 1 & 1 \\ 2 & 5 & 5 \\ 8 & 7 & 7 \end{pmatrix} = \underline{0}$ since 2 columns are equal,

d) $\mathrm{Det}\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 5 \end{pmatrix} = 1\,\mathrm{Det}\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} - 3\,\mathrm{Det}\begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}$

$= 5\,\mathrm{Det}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} - 6\,\mathrm{Det}\begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$

$= 5 - 18 = \underline{-13}$.