Homework 6

Exercise 2

From H 1, ex 5:

a) $V_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, basis $= \{V_1\}$.

b) $V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $V_2 = \frac{1}{c}\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right) = \frac{1}{c} \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Basis $\{V_1, V_2\}$.

c) $V_1 = \frac{1}{c} \begin{pmatrix} \pi - 7 \\ 2\pi + 4 \\ 18 \end{pmatrix}$ with $c^2 = (\pi - 7)^2 + (2\pi + 4)^2 + 18^2$

$= \pi^2 - 14\pi + 49 + 4\pi^2 + 16\pi + 16 + 324$

$= 5\pi^2 + 2\pi + 389$

$\Rightarrow V_1 = \frac{1}{\sqrt{5\pi^2 + 2\pi + 389}} \begin{pmatrix} \pi - 7 \\ 2\pi + 4 \\ 18 \end{pmatrix}$, basis $= \{V_1\}$.

d) No basis since it is a subspace of dimension 0.

Exercise 1

i) Let $f, g, h \in C([0,1])$ and $\lambda \in \mathbb{R}$. One has

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx = \int_0^1 g(x) f(x)\, dx = \langle g, f \rangle$,

$\langle f+g, h \rangle = \int_0^1 (f(x) + g(x)) h(x)\, dx = \int_0^1 f(x) h(x)\, dx + \int_0^1 g(x) h(x)\, dx$

$= \langle f, h \rangle + \langle g, h \rangle$,

$\langle \lambda f, g \rangle = \int_0^1 \lambda f(x) g(x)\, dx = \lambda \int_0^1 f(x) g(x)\, dx = \lambda \langle f, g \rangle$

$\Vert \int_0^1 f(x) \lambda g(x)\, dx = \langle f, \lambda g \rangle$.

$\langle f, f \rangle = \int_0^1 f(x)^2\, dx$ which is equal to 0 if and only if $f = 0$. Thus $\langle \cdot, \cdot \rangle$ defines a scalar product on $C([0,1])$.

ii) Let $\underline{\underline{v_1(x) = 1}}$ with $\|v_1\|^2 = \int_0^1 1 \cdot 1 \, dx = 1$

Then $v_2(x) = \frac{1}{c}\left(x - \left(\int_0^1 x \cdot 1 \, dx \right) 1 \right) = \frac{1}{c}\left(x - \frac{1}{2}\right)$

and c is chosen such that $\|v_2\| = 1$, i.e.

$$\|v_2\|^2 = \frac{1}{c^2} \int_0^1 (x - \tfrac{1}{2})^2 dx = \frac{1}{c^2} \int_0^1 (x^2 - x + \tfrac{1}{4}) \, dx$$

$$= \frac{1}{c^2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4} \right) = \frac{1}{c^2} \frac{1}{12}(4 - 6 + 3) = \frac{1}{c^2} \frac{1}{12} .$$

Thus $\frac{1}{c^2} = 12 \Rightarrow \frac{1}{c} = \sqrt{12}$. $\underline{\underline{v_2(x) = \sqrt{12}\left(x - \frac{1}{2}\right)}}$.

Finally, $v_3(x) = \frac{1}{c}\left(x^2 - \left(\int_0^1 x^2 \cdot 1 \, dx \right) 1 - 12 \left(\int_0^1 x^2 (x - \tfrac{1}{2}) \, dx \right) (x - \tfrac{1}{2}) \right)$

$$= \frac{1}{c}\left(x^2 - \frac{1}{3} - 12\left(\frac{1}{4} - \frac{1}{6} \right)(x - \tfrac{1}{2}) \right) = \frac{1}{c}\left(x^2 - x + \frac{1}{6} \right) .$$

And $\|v_2\|^2 = 1 \Leftrightarrow$

$$\frac{1}{c^2} \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{1}{3}x^2 - \frac{1}{3}x \right) dx = 1$$

$$\Leftrightarrow \frac{1}{c^2}\left(\frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{2}{4} + \frac{1}{9} - \frac{1}{6} \right) = 1$$

$$\Leftrightarrow \frac{1}{c^2} \frac{36 + 60 + 5 - 90 + 20 - 30}{36 \cdot 5} = \frac{1}{c^2} \frac{1}{180} = 1$$

$$\Leftrightarrow \frac{1}{c^2} = 180 = 36 \cdot 5 \Rightarrow \frac{1}{c} = 6\sqrt{5} .$$

$$\Rightarrow \underline{\underline{v_3(x) = 6\sqrt{5}\left(x^2 - x + \frac{1}{6} \right)}} .$$

Exercise 3

i) Let $X_1, X_2, X, Y_1, Y_2, Y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, then

- $F_A(X_1+X_2, Y) = {}^t(X_1+X_2) A Y = {}^tX_1 A Y + {}^tX_2 A Y = F_A(X_1,Y) + F_A(X_2,Y)$
- $F_A(\lambda X, Y) = {}^t(\lambda X) A Y = \lambda {}^tX A Y = \lambda F_A(X,Y)$
- $F_A(X, Y_1+Y_2) = {}^tX A (Y_1+Y_2) = {}^tX A Y_1 + {}^tX A Y_2 = F_A(X,Y_1) + F_A(X,Y_2)$
- $F_A(X, \lambda Y) = {}^tX A(\lambda Y) = \lambda {}^tX A Y = \lambda F_A(X,Y)$.

Thus, F_A is a bilinear map.

ii) $F_A(X,Y) = {}^tX A Y = X \cdot (AY) = \sum_{i=1}^n x_i (AY)_i$

$= \sum_{i=1}^n x_i \sum_{j=1}^n a_{ij} y_j = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i y_j \underset{\uparrow}{=} \sum_{i=1}^n \sum_{j=1}^n a_{ji} x_i y_j$

$a_{ij} = a_{ji}$ since A symmetric

$= \sum_{j=1}^n y_j \sum_{i=1}^n a_{ji} x_i = \sum_{j=1}^n y_j (AX)_j = Y \cdot (AX) = {}^tY A X = F_A(Y,X)$.

iii) From i) and ii), F_A satisfies the first 3 conditions of a scalar product. It remains to impose the condition 4, i.e. $F_A(X,X) \geq 0$ (and equality iff $X=0$)

$\Leftrightarrow \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \geq 0$ with equality iff $x_j = 0 \ \forall j = 1, \dots, n$.

iv) • Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then $(x_1, x_2) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix}$

$= x_1^2 + 4x_1x_2 + x_2^2 = (x_1+x_2)^2 + 2x_1x_2$. For example for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, one has $F_A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = -2 \not\geq 0$.

Thus A does not generate a scalar product.

• If $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, then $(x_1 \ x_2 \ x_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$

$= x_1^2 + (x_1-x_2)^2 + x_2^2 + (x_2-x_3)^2 + x_3^2 \geq 0$, and this expression is equal to 0 iff $x_1 = x_2 = x_3 = 0$, i.e. iff $X = 0$.

Thus A generates a scalar product.

Exercise 4

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and recall that

$$X \wedge Y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$ Then one sees that

$(X + X') \wedge Y = X \wedge Y + X' \wedge Y$ and that

$X \wedge (Y + Y') = X \wedge Y + X \wedge Y'$. Similarly

$(\lambda X) \wedge Y = \lambda (X \wedge Y)$ and $X \wedge (\lambda Y) = \lambda (X \wedge Y)$

for all $\lambda \in \mathbb{R}$. Thus, the cross product is bilinear. In addition, one observes from the definition that $X \wedge X = 0$, and thus the cross product is alternating.

Exercise 5

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$, and $Y \in \mathbb{R}^3$. Then

$F_1(X, Y) := {}^tX \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} Y$ is an alternating bilinear map

since $F_1(X, X) = (x_1\ x_2\ x_3) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1\ x_2\ x_3) \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}$

$$= x_1 x_3 - x_3 x_1 = 0.$$

The same holds for $F_2(X, Y) := {}^tX \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$

and for $F_3(X, Y) := {}^tX \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} Y$.