Homework 6

Exercise 1 *We consider the real vector space $V := C([0,1])$ made of continuous real functions on $[0,1]$ and endow it with the map*

$$V \times V \ni (f,g) \mapsto \langle f,g\rangle := \int_0^1 f(x)\,g(x)\,\mathrm{d}x \ \in \mathbb{R}.$$

Show that

(i) $\langle\cdot,\cdot\rangle$ *is a scalar product on* V,

(ii) If W is the subspace of V generated by the three functions $x \mapsto 1$ (constant function), $x \mapsto x$ (identity function), and $x \mapsto x^2$, find an orthonormal basis for W.

Exercise 2 *Find an orthonormal basis for the space of solutions of the following systems:*

$$a)\ \begin{cases} 2x+y-z &= 0 \\ 2x+y+z &= 0 \end{cases} \quad b)\ \begin{cases} x-y+z &= 0 \end{cases} \quad c)\ \begin{cases} 4x+7y-\pi z &= 0 \\ 2x-y+z &= 0 \end{cases}$$

$$d)\ \begin{cases} x+y+z &= 0 \\ x-y &= 0 \\ y+z &= 0 \end{cases}$$

Exercise 3 *For any symmetric matrix $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{R})$, we define the map*

$$\mathrm{F}_{\mathcal{A}} : \mathbb{R}^n \times \mathbb{R}^n \ni (X,Y) \mapsto \mathrm{F}_{\mathcal{A}}(X,Y) := {}^tX\mathcal{A}Y \in \mathbb{R}.$$

(i) Show that $\mathrm{F}_{\mathcal{A}}$ is a bilinear map,

(ii) Show that $\mathrm{F}_{\mathcal{A}}(X,Y) = \mathrm{F}_{\mathcal{A}}(Y,X)$ for any $X,Y \in \mathbb{R}^n$.

(iii) When does $\mathrm{F}_{\mathcal{A}}$ define a scalar product ?

(iv) If $\mathcal{A}$ is one of the following matrices, does $\mathrm{F}_{\mathcal{A}}$ define a scalar product ?

$$\mathcal{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Exercise 4 *Show that the the cross product in $\mathbb{R}^3$ is a bilinear alternating map.*

Exercise 5 *Exhibit 3 different alternating bilinear maps on $\mathbb{R}^3$.*