Exercise 1

i) $\|\lambda X\|^2 = \langle \lambda X, \lambda X\rangle = \lambda^2 \langle X, X\rangle = \lambda^2 \|X\|^2 \Rightarrow \|\lambda X\| = |\lambda| \, \|X\|$.

ii) $\|X+Y\|^2 = \langle X+Y, X+Y\rangle = \langle X,X\rangle + \langle X,Y\rangle + \langle Y,X\rangle + \langle Y,Y\rangle$

$$= \|X\|^2 + \|Y\|^2 + 2\langle X,Y\rangle.$$

Thus, $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$ if and only if $\langle X,Y\rangle = 0$, i.e. if and only if $X \perp Y$.

iii) $\|X+Y\|^2 + \|X-Y\|^2 = \|X\|^2 + \|Y\|^2 + 2\langle X,Y\rangle + \|X\|^2 + \|Y\|^2 - 2\langle X,Y\rangle$

$$= 2\|X\|^2 + 2\|Y\|^2.$$

iv) One has $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2 + 2\langle X,Y\rangle$

$$\leq \|X\|^2 + \|Y\|^2 + 2|\langle X,Y\rangle|$$

$$\leq \|X\|^2 + \|Y\|^2 + 2\|X\|\,\|Y\|$$

$$= (\|X\| + \|Y\|)^2.$$

$\Rightarrow \|X+Y\| \leq \|X\| + \|Y\|$.

Exercise 2

i) $\mathrm{Tr}(A+B) = \sum_{j=1}^n (a_{jj} + b_{jj}) = \sum_{j=1}^n a_{jj} + \sum_{j=1}^n b_{jj} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$,

$\mathrm{Tr}(\lambda A) = \sum_{j=1}^n (\lambda a_{jj}) = \lambda \sum_{j=1}^n a_{jj} = \lambda \, \mathrm{Tr}(A)$,

thus $\mathrm{Tr} : M_n(\mathbb{R}) \to \mathbb{R}$ is linear.

ii) $\mathrm{Tr}(AB) = \sum_{j=1}^n (AB)_{jj} = \sum_{j=1}^n \left(\sum_{k=1}^n a_{jk} b_{kj}\right) = \sum_{j=1}^n \sum_{k=1}^n b_{kj} a_{jk}$

$$= \sum_{k=1}^n \sum_{j=1}^n b_{kj} a_{jk} = \sum_{k=1}^n (BA)_{kk} = \mathrm{Tr}(BA).$$

iii) From ii), $\mathrm{Tr}(C^{-1}(AC)) = \mathrm{Tr}((AC)C^{-1}) = \mathrm{Tr}(A)$.

iv) Let us set $\langle A,B\rangle := \mathrm{Tr}(AB)$. By ii), $\langle A,B\rangle = \langle B,A\rangle$; by i), $\langle A+B, C\rangle = \mathrm{Tr}((A+B)C) = \mathrm{Tr}(AC+BC) = \mathrm{Tr}(AC) + \mathrm{Tr}(BC)$ $= \langle A,C\rangle + \langle B,C\rangle$.

By i), $\langle \lambda A, B\rangle = \mathrm{Tr}(\lambda AB) = \lambda\, \mathrm{Tr}(AB) = \lambda \langle A, B\rangle$

and $\langle \lambda A, B\rangle = \mathrm{Tr}(\lambda AB) = \mathrm{Tr}(A(\lambda B)) = \langle A, \lambda B\rangle$.

Finally, $\langle A, A\rangle = \mathrm{Tr}(AA) = \sum_{j=1}^{n} (AA)_{jj} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\, a_{kj}$

$$\underset{\uparrow \text{ if } A = {}^{T}\!A \iff a_{jk} = a_{kj}}{=} \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\, a_{jk} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}^2$$

Thus, $\langle A, A\rangle = 0$ if and only if $a_{jk} = 0 \quad \forall j,k$

i.e. if and only if $A = 0$.

We have verified the condition for $\langle \cdot, \cdot \rangle$ to be a scalar product.

Exercise 3

We will choose c such that $\| v_2 \| = 1$

Let $v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$, $v_2 = \frac{1}{c}\left[\begin{pmatrix}1\\0\\-1\\2\end{pmatrix} - \left(\begin{pmatrix}1\\0\\-1\\2\end{pmatrix}\cdot v_1\right) v_1\right]$

$$= \frac{1}{c}\left[\begin{pmatrix}1\\0\\-1\\2\end{pmatrix} - \frac{1}{\sqrt{3}}(1+2)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right] = \frac{1}{c}\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix},$$

$$v_3 = \frac{1}{d}\left[\begin{pmatrix}1\\-2\\0\\0\end{pmatrix} - \frac{1}{3}(-1)\begin{pmatrix}1\\1\\0\\1\end{pmatrix} - \frac{1}{3}(2)\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix}\right] = \frac{1}{d}\left[\begin{pmatrix}1\\-2\\0\\0\end{pmatrix} + \begin{pmatrix}1/3\\1/3\\0\\1/3\end{pmatrix} + \begin{pmatrix}0\\2/3\\2/3\\-2/3\end{pmatrix}\right]$$

$$= \frac{1}{d}\begin{pmatrix}4/3\\-1\\2/3\\-1/3\end{pmatrix} = \frac{1}{3d}\begin{pmatrix}4\\-3\\2\\-1\end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix}4\\-3\\2\\-1\end{pmatrix}.$$