Nagoya University, G30 program Spring 2015

Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 5

Exercise 1 *Let V be a real vector space endowed with a scalar product. Prove the following relations for $X, Y \in V$ and $\lambda \in \mathbb{R}$:*

(i) $\|\lambda X\| = |\lambda|\,\|X\|$,

(ii) $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$ *if and only if* $X \perp Y$,

(iii) $\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2$,

(iv) $\|X+Y\| \leq \|X\| + \|Y\|$.

Exercise 2 *Let* $\mathcal{A} = (a_{jk}) \in M_n(\mathbb{R})$ *and define* $\mathsf{Tr}(\mathcal{A}) = \sum_{j=1}^n a_{jj}$, *where* $\mathsf{Tr}(\mathcal{A})$ *is called* the trace of $\mathcal{A}$. *Show the following properties:*

(i) $\mathsf{Tr} : M_n(\mathbb{R}) \to \mathbb{R}$ *is a linear map,*

(ii) $\mathsf{Tr}(\mathcal{AB}) = \mathsf{Tr}(\mathcal{BA})$, *for any* $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{R})$,

(iii) If $\mathcal{C} \in M_n(\mathbb{R})$ *is an invertible matrix, then* $\mathsf{Tr}(\mathcal{C}^{-1}\mathcal{AC}) = \mathsf{Tr}(\mathcal{A})$,

(iv) If $M_n^s(\mathbb{R})$ *denotes the vector space of all* $n \times n$ *symmetric matrices, then the map*

$$\mathsf{Tr} : M_n^s(\mathbb{R}) \times M_n^s(\mathbb{R}) \ni (\mathcal{A}, \mathcal{B}) \mapsto \mathsf{Tr}(\mathcal{AB}) \in \mathbb{R}$$

defines a scalar product on $M_n^s(\mathbb{R})$. *We recall that a matrix* $\mathcal{A}$ *is symmetric if* $\mathcal{A} = {}^t\mathcal{A}$.

Exercise 3 *Find an orthonormal basis for the subspace of* $\mathbb{R}^4$ *defined by the three vectors* $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$.