Exercise 1

Computation with block matrices

$$\left(\begin{array}{c:c} A_{11} & A_{12} \end{array}\right) \left(\begin{array}{c:c} B_{11} & B_{12} \\ \hdashline B_{21} & B_{22} \end{array}\right)$$

$$= \left(\begin{array}{c:c} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \end{array}\right)$$

$$= \left(\begin{array}{c:c} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} 7 & 8 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} 9 \end{pmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c:c} \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} -7 & -8 \\ 7 & 8 \end{pmatrix} & \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} -9 \\ 9 \end{pmatrix} \end{array}\right)$$

$$= \underline{\underline{\left(\begin{array}{cc:c} -3 & -3 & -3 \\ 8 & 10 & 12 \end{array}\right)}} .$$

Otherwise $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -3 & -3 & -3 \\ 8 & 10 & 12 \end{pmatrix}}}$.

Exercise 2

Assume $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ is an inverse for A, i.e.

$$AB = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

$$= \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Thus one obtains the conditions

$$\begin{cases} A_{22}B_{22} = \mathbb{1} \\ A_{22}B_{21} = 0 \\ A_{11}B_{11} + A_{12}B_{21} = \mathbb{1} \\ A_{11}B_{12} + A_{12}B_{22} = 0 \end{cases} \Rightarrow \begin{cases} B_{22} = A_{22}^{-1} \\ B_{21} = 0 \quad \text{since } A_{22} \text{ is invertible} \\ B_{11} = A_{11}^{-1} \quad \text{since } B_{21} = 0 \\ A_{11}B_{12} = -A_{12}B_{22} \end{cases}$$

Therefore, A is invertible if and only if A_{11}, A_{22} are invertible, and in this case its inverse B satisfies

$$\begin{cases} B_{11} = A_{11}^{-1} \\ B_{22} = A_{22}^{-1} \\ B_{12} = -A_{11}^{-1}A_{12}B_{22} = -A_{11}^{-1}A_{12}A_{22}^{-1} \end{cases}$$

Note that there is no condition on A_{12} for the invertibility of A.

Exercise 3

Recall that L_A is <u>injective</u> if $L_A(X) \neq 0$ whenever $X \neq 0$, or equivalently $AX \neq 0$ whenever $X \neq 0$, or also $AX \neq AY$ whenever $X \neq Y$.

Since AX admits only one preimage, one sets $B(AX) = X$ (B gives the preimage of AX), and one has $X = B(AX) = (BA)X \iff BA = 1\!\!1_n$.

Now if L_A is surjective, then for any $Y \in \mathbb{F}^n$, there exists at least one $X \in \mathbb{F}^n$ such that $AX = Y$.

Thus one can set $CY = X$ and one has

$Y = AX = ACY \iff AC = 1\!\!1_n$.

In summary, L_A injective $\iff BA = 1\!\!1_n$ for some $B \in M_n(\mathbb{F})$, and L_A surjective $\iff AC = 1\!\!1_n$ for some $C \in M_n(\mathbb{F})$.

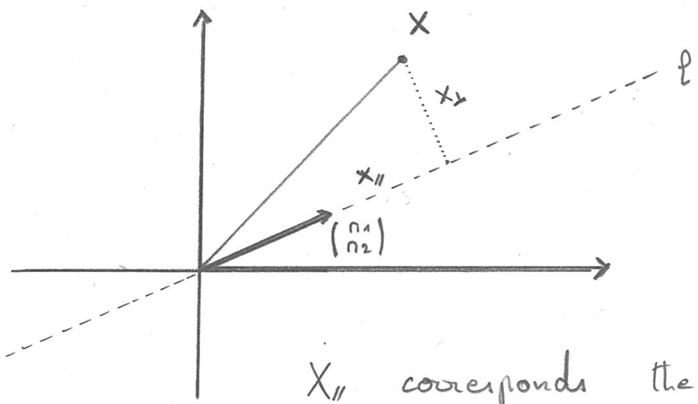
Since we have seen that $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is injective if and only if L_A is surjective, it follows that $BA = 1\!\!1_n$ if and only if $AC = 1\!\!1_n$.

It then follows that

$$B = B\,1\!\!1_n = B(AC) = (BA)C = 1\!\!1_n\,C = C.$$

And finally $BA = 1\!\!1_n = AB$, which means that B is the inverse of A, i.e. $B = A^{-1}$.

Exercise 4



From what we have seen during the first semester, $X_{/\!/} = c\binom{n_1}{n_2}$ with c given by the expression $c = \dfrac{X \cdot N}{\|N\|^2}$. $X_{/\!/}$ corresponds the orthogonal projection of X along N. Thus, if $X = \binom{x_1}{x_2}$ one has

$$X_{/\!/} = \frac{X \cdot N}{\|N\|^2}\binom{n_1}{n_2} = \frac{x_1 n_1 + x_2 n_2}{1}\binom{n_1}{n_2} = \begin{pmatrix} x_1 n_1^2 + x_2 n_1 n_2 \\ x_1 n_1 n_2 + x_2 n_2^2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}\,.$$

Let us set $P := \begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix}$ and check that P is a projection:

$$P^2 = \begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix}\begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix} = \begin{pmatrix} n_1^4 + n_1^2 n_2^2 & n_1^3 n_2 + n_1 n_2^3 \\ n_1^3 n_2 + n_1 n_2^3 & n_1^2 n_2^2 + n_2^4 \end{pmatrix}$$

$$= \begin{pmatrix} n_1^2 (n_1^2 + n_2^2) & n_1 n_2 (n_1^2 + n_2^2) \\ n_1 n_2 (n_1^2 + n_2^2) & n_2^2 (n_1^2 + n_2^2) \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} n_1^2 & n_1 n_2 \\ n_1 n_2 & n_2^2 \end{pmatrix} = P.$$

because $n_1^2 + n_2^2 = \|N\|^2 = 1$.

Therefore, P is a projection.

Exercise 5

1) Similarly, $$X_{\parallel} = \frac{X \cdot N}{\|N\|^2} N = \frac{x_1 n_1 + x_2 n_2 + x_3 n_3}{1} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix}}_{\text{matrix } P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

P is a projection since $$P^2 = \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix} \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} n_1^4 + n_1^2 n_2^2 + n_1^2 n_3^2 & n_1^3 n_2 + n_1 n_2^3 + n_1 n_2 n_3^2 & n_1^3 n_3 + n_1 n_2^2 n_3 + n_1 n_3^3 \\ n_1^3 n_2 + n_1 n_2^3 + n_1 n_2 n_3^2 & n_1^2 n_2^2 + n_2^4 + n_2^2 n_3^2 & n_1^2 n_2 n_3 + n_2^3 n_3 + n_2 n_3^3 \\ n_1^3 n_3 + n_1 n_2^2 n_3 + n_1 n_3^3 & n_1^2 n_2 n_3 + n_2^3 n_3 + n_2 n_3^3 & n_1^2 n_3^2 + n_2^2 n_3^2 + n_3^4 \end{pmatrix}$$

$$= (n_1^2 + n_2^2 + n_3^2) \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{pmatrix} = P .$$

2) Since $X = X_{\parallel} + X_{\perp} \Rightarrow X_{\perp} = X - X_{\parallel} = X - PX = (1-P)X.$

Thus $X_{\perp} = (1-P)X$, with $Q := (1-P)$ also a projection.

Let us observe that $$(PX) \overset{\text{scalar product}}{\cdot} N = (x_1 n_1 + x_2 n_2 + x_3 n_3) \underbrace{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}_{= n_1^2 + n_2^2 + n_3^2 = 1}$$

$$= x_1 n_1 + x_2 n_2 + x_3 n_3 = X \cdot N = (1X) \cdot N .$$

Therefore $(PX) \cdot N = (1X) \cdot N \iff [(1-P)X] \cdot N = 0 \iff QX \cdot N = 0$ $\iff X_{\perp} \cdot N = 0$. By definition of $H_{0,N}$, it means that the point $X_{\perp}$ (corresponding to the ending point of the vector $X_{\perp}$) belongs to $H_{0,N}$.