Nagoya University, G30 program Spring 2015
Mathematics Tutorial II (linear algebra part) Instructor : Serge Richard

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Homework 4

Exercise 1 *Block matrices are matrices which are partitioned into rectangular submatrices called* blocks. *For example, let $\mathcal{A} \in M_{n+m}(\mathbb{R})$ be the block matrix*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}$$

with $\mathcal{A}_{11} \in M_n(\mathbb{R})$, $\mathcal{A}_{22}(\mathbb{R}) \in M_m(\mathbb{R})$, $\mathcal{A}_{12} \in M_{n\times m}(\mathbb{R})$, and $\mathcal{A}_{21} \in M_{m\times n}(\mathbb{R})$. Such matrices can be multiplied as if every blocks where scalars (with the usual multiplication of matrices), as long as the products are well defined. For example, check this statement by computing the product $\mathcal{AB}$ in two different ways with the following matrices: $\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \end{pmatrix}$ with $\mathcal{A}_{11} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathcal{A}_{12} = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$, and $\mathcal{B} = \left(\begin{smallmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{smallmatrix}\right)$ with $\mathcal{B}_{11} = \left(\begin{smallmatrix} 1 & 2 \\ 4 & 5 \end{smallmatrix}\right)$, $\mathcal{B}_{12} = \left(\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right)$, $\mathcal{B}_{21} = \begin{pmatrix} 7 & 8 \end{pmatrix}$, and $\mathcal{B}_{22} = \begin{pmatrix} 9 \end{pmatrix}$.

Exercise 2 *Let $\mathcal{A} \in M_{n+m}(\mathbb{R})$ be the block matrix*

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{O} & \mathcal{A}_{22} \end{pmatrix}$$

with $\mathcal{A}_{11} \in M_n(\mathbb{R})$, $\mathcal{A}_{22}(\mathbb{R}) \in M_m(\mathbb{R})$ and $\mathcal{A}_{12} \in M_{n\times m}(\mathbb{R})$.

(i) For which choice of $\mathcal{A}_{11}$, $\mathcal{A}_{12}$ and $\mathcal{A}_{22}$ is $\mathcal{A}$ invertible ?

(ii) If $\mathcal{A}$ is invertible, what is $\mathcal{A}^{-1}$, in terms of $\mathcal{A}_{11}$, $\mathcal{A}_{12}$ and $\mathcal{A}_{22}$?

Exercise 3 *Show that for any $\mathcal{A} \in M_n(\mathbb{F})$, the following statements are equivalent:*

(i) There exists $\mathcal{B} \in M_n(\mathbb{F})$ such that $\mathcal{BA} = \mathbf{1}_n$,

(ii) There exists $\mathcal{C} \in M_n(\mathbb{F})$ such that $\mathcal{AC} = \mathbf{1}_n$.

In addition, whenever (i) or (ii) holds, then $\mathcal{B} = \mathcal{C}$, and $\mathcal{A}$ is invertible with $\mathcal{A}^{-1} = \mathcal{B} = \mathcal{C}$.

Exercise 4 *Let $N = \left(\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right)$ be a vector in $\mathbb{R}^2$ with $\|N\| = 1$, and let ℓ be the line in $\mathbb{R}^2$ passing trough $\mathbf{0} \in \mathbb{R}^2$ and parallel to N. Then any vector $X \in \mathbb{R}^2$ can be written uniquely as $X = X_\parallel + X_\perp$, where $X_\parallel$ is a vector parallel to ℓ and $X_\perp$ is a vector perpendicular to ℓ. Show that there exists a projection $\mathrm{P} \in M_2(\mathbb{R})$ such that $X_\parallel = \mathrm{P}\,X$, and express P in terms of n_1 and n_2.*

Exercise 5 *1) Do the same exercise in $\mathbb{R}^3$ with N given by $\left(\begin{smallmatrix} n_1 \\ n_2 \\ n_3 \end{smallmatrix}\right)$.*

2) Show that there also exists a projection Q such that $X_\perp = \mathrm{Q}\,X$. If $H_{\mathbf{0},N}$ is the plane passing through $\mathbf{0} \in \mathbb{R}^3$ and perpendicular to N, show that $X_\perp \in H_{\mathbf{0},N}$.