Homework 3

Exercise 1

i) $(1-P)^2 = (1-P)\circ(1-P) = (1-P)\circ 1 - (1-P)\circ P$
$= 1-P - P + P^2 = 1-P$ since $P^2 = P$.

Thus $1-P$ is also a projection.

In addition, $(1-P)P = P - P^2 = 0$ and $P(1-P) = P-P^2 = 0$.

ii) Observe that from $P(1-P) = 0$, one infers that $\mathrm{Ran}(1-P) \subset \mathrm{Ker}(P)$. On the other hand, if $x \in \mathrm{Ker}(P)$ it follows that $x = 1x + 0 = (1-P)x$, which implies that $x \in \mathrm{Ran}(1-P)$. One thus infers that $\mathrm{Ker}(P) \subset \mathrm{Ran}(1-P)$, and then $\underline{\mathrm{Ker}(P) = \mathrm{Ran}(1-P)}$.

As a consequence, for any $y \in V$ one has

$$y = (1-P+P)y = (1-P)y + Py \in \mathrm{Ran}(1-P) + \mathrm{Ran}(P)$$
$$= \mathrm{Ker}(P) + \mathrm{Ran}(P).$$

In other words, any element y of V is the sum of an element of $\mathrm{Ker}(P)$ and an element of $\mathrm{Ran}(P)$.

iii) Assume $x \in \mathrm{Ker}(P) \cap \mathrm{Ran}(P)$. Since $x \in \mathrm{Ran}(P)$, $\exists y \in V$ such that $x = Py$. One thus infers that

$$\underset{\substack{\uparrow \\ \text{because } x \in \mathrm{Ker}(P)}}{0 =} Px = P(Py) = P^2y \underset{\substack{\uparrow \\ \text{because } P \text{ is a projection}}}{=} Py = x$$

and thus $x = 0$. It means that the only element in $\mathrm{Ker}(P)$ and $\mathrm{Ran}(P)$ is 0.

Exercise 2

1) One has $L = L_A$ with $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in M_2(\mathbb{R})$.

For finding the inverse of A, one has

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right).$$

Thus, the inverse of L is given by $L_{A^{-1}}$ with $A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$.

2) By the same method one finds that $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} -1 & 3 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$. Thus, L is invertible, with $L^{-1} = L_{A^{-1}}$.

Exercise 3

One easily observes that $(G \circ F) \circ (F^{-1} \circ G^{-1}) = G \circ \mathbb{1} \circ G^{-1} = \mathbb{1}$ and that $(F^{-1} \circ G^{-1}) \circ (G \circ F) = F^{-1} \circ \mathbb{1} \circ F = \mathbb{1}$.

Since the inverse of a linear map is unique (if it exists), one infers that $G \circ F$ is invertible, with

$$\underline{(G \circ F)^{-1} = F^{-1} \circ G^{-1}}.$$

Exercise 4

Let $\mathcal{V} = \{V_1, \dots, V_n\}$ be a basis of V and let $\mathcal{V}' = \{V_1', \dots, V_n'\}$ be a second basis of V (V can be $\mathbb{R}^n$, but the same proof holds for arbitrary vector spaces of finite dimension).

Since $\mathcal{V}$ and $\mathcal{V}'$ are basis, there B and C belonging to $M_n(\mathbb{F})$ ($\mathbb{F}$ can be $\mathbb{R}$ but can be a more general field) such that $X = BX'$ and $X' = CX$, with $X^{(\prime)}$ the coordinate vector of any x with respect to the basis $\mathcal{V}^{(\prime)}$, cf Lecture notes.

It follows that $X = BX' = BCX$ and $X' = CX = CBX'$, from which one deduces that $BC = \mathbb{1}$ and $CB = \mathbb{1}$.

This means that B is invertible, with inverse C, or equivalently C is invertible, with inverse B.

Exercice 5

i) $F((x_1, x_2, \dots) + (x_1', x_2', \dots))$

$= F(x_1 + x_1', x_2 + x_2', \dots)$

$= (0, x_1 + x_1', x_2 + x_2', \dots)$

$= (0, x_1, x_2, \dots) + (0, x_1', x_2', \dots)$

$= F(x_1, x_2, \dots) + F(x_1', x_2', \dots)$.

Similarly, $F(\lambda(x_1, x_2, \dots)) = F(\lambda x_1, \lambda x_2, \dots)$

$= (0, \lambda x_1, \lambda x_2, \dots) = \lambda (0, x_1, x_2, \dots) = \lambda F(x_1, x_2, \dots)$.

Thus F is a linear map.

ii) Observe that $F(x_1, x_2, \dots) = (0, 0, \dots) \Leftrightarrow$

$(0, x_1, x_2, \dots) = (0, 0, \dots) \Leftrightarrow x_1 = 0 = x_2 = x_3 \dots$

Thus, $\mathrm{Ker}(F) = (0, 0, \dots)$, and F is injective.

iii) F is not surjective since $(c, 0, 0 \dots) \notin \mathrm{Ran}(F)$ for any $c \neq 0$.

iv) Let us set $G : V \to V$, $G(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

It is easy to check that G is linear, and one has

$(G \circ F)(x_1, x_2, \dots) = G(F(x_1, x_2, \dots)) = G(0, x_1, x_2, \dots)$

$= (x_1, x_2, \dots) \Rightarrow G \circ F = \mathbb{1}$.

v) On the other hand $(F \circ G)(x_1, x_2, \dots) = F(G(x_1, x_2, \dots))$

$= F(x_2, x_3, \dots) = (0, x_2, x_3, \dots) \neq (x_1, x_2, x_3, \dots)$

$\Rightarrow F \circ G \neq \mathbb{1}$. The information on x_1 is lost by $F \circ G$.

vi) When V is of finite dimension, we know that

F injective $\Leftrightarrow$ F bijective $\Leftrightarrow$ F invertible.

⚠ If V is of infinite dimension, F injective $\not\Rightarrow$ F invertible.