Homework 3

Exercise 1 *Let* V *be a real vector space, and let* $\mathrm{P}: V \to V$ *be a linear map satisfying* $\mathrm{P}^2 = \mathrm{P}$*. Such a linear map is called* a projection.

(i) Show that $\mathbf{1} - \mathrm{P}$ *is also a projection, and that* $(\mathbf{1} - \mathrm{P})\mathrm{P} = \mathrm{P}(\mathbf{1} - \mathrm{P}) = \mathbf{0}$,

(ii) Show that $V = \mathsf{Ker}(\mathrm{P}) + \mathsf{Ran}(\mathrm{P})$,

(iii) Show that the intersection of $\mathsf{Ker}\, P$ *and* $\mathsf{Ran}(\mathrm{P})$ *is* $\{\mathbf{0}\}$.

Exercise 2 *Let* $\mathrm{L}: \mathbb{R}^2 \to \mathbb{R}^2$ *be the linear map defined by* $\mathrm{L}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+y \\ x-y \end{smallmatrix}\right)$*. Show that* L *is invertible and find its inverse. Same question with the map* $\mathrm{L}: \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $\mathrm{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ x+z \\ x+y+3z \end{pmatrix}$.

Exercise 3 *Let* F, G *be invertible linear maps from a vector space into itself. Show that*

$$(\mathrm{G} \circ \mathrm{F})^{-1} = \mathrm{F}^{-1} \circ \mathrm{G}^{-1}.$$

Exercise 4 *Show that the matrix* $\mathcal{B}: \mathbb{R}^n \to \mathbb{R}^n$ *defining a change of basis in* $\mathbb{R}^n$ *is always invertible.*

Exercise 5 *Let* V *be the set of all infinite sequences of real numbers* $(x_1, x_2, x_3, \dots)$*. We endow* V *with the pointwise addition and multiplication, i.e.*

$$(x_1, x_2, x_3, \dots) + (x_1', x_2', x_3', \dots) = (x_1 + x_1', x_2 + x_2', x_3 + x_3', \dots)$$

and $\lambda(x_1, x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$*, which make* V *an infinite dimensional vector space.*

Define the map $\mathrm{F}: V \to V$*, called* shift operator*, by*

$$\mathrm{F}(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

(i) Is F *a linear map ?*

(ii) Is F *injective, and what is the kernel of* F *?*

(iii) Is F *surjective ?*

(iv) Show that there is a linear map $\mathrm{G}: V \to V$ *such that* $\mathrm{G} \circ \mathrm{F} = \mathbf{1}$.

(v) Does the map G *have the property that* $\mathrm{F} \circ \mathrm{G} = \mathbf{1}$ *?*

(vi) What is different from the finite dimensional case, i.e. when V *is of finite dimension ?*