Homework 2

Exercise 1

1) Let $A, B \in M_n(\mathbb{R})$, then $P(A+B) = \frac{1}{2}\{(A+B) + {}^t(A+B)\}$

$$= \frac{1}{2}\{A + B + {}^tA + {}^tB\} = \frac{1}{2}\{A + {}^tA\} + \frac{1}{2}\{B + {}^tB\} = P(A) + P(B).$$

In addition, $P(\lambda A) = \frac{1}{2}\{\lambda A + {}^t(\lambda A)\} = \frac{1}{2}\{\lambda(A + {}^tA)\}$

$$= \lambda \frac{1}{2}\{A + {}^tA\} = \lambda P(A).$$

Thus, P is a linear map.

2) $\mathrm{Ker}(P) = \{A \in M_n(\mathbb{R}) \mid P(A) = 0\}$

$$= \{A \in M_n(\mathbb{R}) \mid \tfrac{1}{2}\{A + {}^tA\} = 0\}$$

$$= \{A \in M_n(\mathbb{R}) \mid {}^tA = -A\}.$$

skew-symmetric matrix.

3) For any A, $\frac{1}{2}\{A + {}^tA\}$ is symmetric since ${}^t\left(\frac{1}{2}\{A + {}^tA\}\right) = \frac{1}{2}\{{}^tA + A\} = \frac{1}{2}\{A + {}^tA\}$. Thus $P(A)$ is a symmetric matrix. On the other hand, if B is a symmetric matrix, i.e. if $B = {}^tB$, then $P(B) = B$. Thus any symmetric matrices is the image by P of an element of $M_n(\mathbb{R})$. One thus concludes that $\mathrm{Ran}(P) = \{B \in M_n(\mathbb{R}) \mid B = {}^tB\}$.

4) Since P is linear, it follows that $\mathrm{Ker}(P)$ and $\mathrm{Ran}(P)$ are subspaces of $M_n(\mathbb{R})$.

One has shown last semester that

$\dim\{B \in M_n(\mathbb{R}) \mid B = {}^tB\} = \frac{n^2 - n}{2} + n = \frac{1}{2}n(n+1)$, and

$\dim\{B \in M_n(\mathbb{R}) \mid {}^tB = -B\} = \frac{1}{2}(n^2 - n) = \frac{1}{2}n(n-1).$

Exercice 2

1) $A \sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ rank $A = 2$, and thus $\dim(\operatorname{Ran} L_A) = 2$.

2) Since $\dim(\operatorname{Ker} L_A) + \dim(\operatorname{Ran} L_A) = 4$ one deduces that $\dim(\operatorname{Ker} L_A) = 2$.

One deduces that $X = {}^t(x_1, x_2, x_3, x_4) \in \operatorname{Ker} L_A$ iff

$$\begin{cases} 2x_1 + x_2 - 4x_3 + 3x_4 = 0 \\ x_2 + 3x_3 - 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = -3x_3 + 2x_4 \\ x_1 = -\frac{1}{2}x_2 + 2x_3 - 3/2\, x_4 = \frac{7}{2}x_3 - \frac{5}{2}x_4 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = 7/2\, x_3 - 5/2\, x_4 \\ x_2 = -3x_3 + 2x_4 \\ x_3 \text{ arbitrary} \\ x_4 \text{ arbitrary} \end{cases}$$

A basis for $\operatorname{Ker}(L_A)$ is thus given by $\left\{ \begin{pmatrix} 7/2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

3) One observes that a special solution is provided by $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ since $L_A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$.

Thus, all solutions of $AX = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 7/2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5/2 \\ 2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}.$$

Exercice 3

a) $\mathcal{F} = \begin{pmatrix} F(E_1) & F(E_2) & F(E_3) \end{pmatrix} = \begin{pmatrix} 1 & -4 & 3 \\ -3 & 2 & 1 \end{pmatrix}$.

b) $\mathcal{F} = \begin{pmatrix} F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 4 & -1 & 5 \end{pmatrix}$.

Exercise 4

Recall that matrix for the change of basis was defined by the relations $V_j = \sum_{i=1}^{3} b_{ij} E_i$. Thus, one has

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which inverse is given by $B^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Since L is represented in the basis $\{E_1, E_2, E_3\}$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} =: \mathcal{L}$

then $\mathcal{L}' = B^{-1} \mathcal{L} B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \underline{\begin{pmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 3/2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$.

Exercise 5

i) Let $x, y \in U$ and $\lambda \in \mathbb{F}$. Then

$(H \circ G)(x+y) = H(G(x+y)) = H(G(x) + G(y)) = H(G(x)) + H(G(y)) = (H \circ G)(x) + (H \circ G)(y)$.

Similarly, $(H \circ G)(\lambda x) = H(G(\lambda x)) = H(\lambda G(x)) = \lambda H(G(x)) = \lambda (H \circ G)(x)$.

Therefore, $\underline{H \circ G \text{ is linear}}$, whenever H and G are linear.

ii) $((H + H') \circ G)(x) = (H + H')(G(x)) = H(G(x)) + H'(G(x)) = (H \circ G)(x) + (H' \circ G)(x)$.

Since x is arbitrary, it follows that $\underline{(H + H') \circ G = H \circ G + H' \circ G}$.

iii) $(H \circ (G + G'))(x) = H((G + G')(x)) = H(G(x) + G'(x)) = H(G(x)) + H(G'(x))$
$= (H \circ G)(x) + (H \circ G')(x)$.

$\Rightarrow \underline{H \circ (G + G') = H \circ G + H \circ G'}$.

iv) $((\lambda H) \circ G)(x) = (\lambda H)(G(x)) = \underbrace{\lambda [H(G(x))]}_{= \lambda (H \circ G)(x)} = H(\lambda G(x)) = \underbrace{H((\lambda G)(x))}_{= (H \circ (\lambda G))(x)}$

$\Rightarrow \underline{(\lambda H) \circ G = \lambda (H \circ G) = H \circ (\lambda G)}$.