Exercises 1 and 2

a) Linearity : for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ in $\mathbb{R}^3$, one has

$$F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) = F\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix} = \begin{pmatrix} x+x' \\ z+z' \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} x' \\ z' \end{pmatrix} = F\begin{pmatrix} x \\ y \\ z \end{pmatrix} + F\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

and for $\lambda \in \mathbb{R}$

$$F\left(\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = F\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda z \end{pmatrix} = \lambda\begin{pmatrix} x \\ z \end{pmatrix} = \lambda F\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Thus, F is linear.

In addition, $\operatorname{Ker}(F) = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$= \left\{\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3 \mid y \text{ is arbitrary}\right\}$,

and $\operatorname{Ran}(F) = \mathbb{R}^2$ since for any $\begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^2$ one has $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \in \mathbb{R}^3$ and $F\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$.

b) Linearity : $F(X+X') = -(X+X') = -X - X' = F(X) + F(X')$ and $F(\lambda X) = -(\lambda X) = \lambda(-X) = \lambda F(X)$.

Thus, F is linear. In addition, $\operatorname{Ker}(F) = \{0\} \subset \mathbb{R}^4$ since $F(X) = 0 \iff -X = 0 \iff X = 0$. Also, $\operatorname{Ran}(F) = \mathbb{R}^4$ since for any $X \in \mathbb{R}^4$, one has $-X \in \mathbb{R}^4$ and $F(-X) = -(-X) = X$.

c) Linearity : F is not linear since $F(\lambda 0) = F(0) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ but $\lambda F(0) = \lambda\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ except if $\lambda = 1$. Since $F(\lambda X) = \lambda F(X)$ should hold for any X and any λ if F were linear, it means that F is not linear.

d) Linearity : $F\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = F\begin{pmatrix} x+x' \\ y+y' \end{pmatrix} = \begin{pmatrix} 2(x+x') \\ (y+y')-(x+x') \end{pmatrix} = \begin{pmatrix} 2x \\ y-x \end{pmatrix} + \begin{pmatrix} 2x' \\ y'-x' \end{pmatrix}$
$= F\begin{pmatrix} x \\ y \end{pmatrix} + F\begin{pmatrix} x' \\ y' \end{pmatrix}$, and $F\left(\lambda\begin{pmatrix} x \\ y \end{pmatrix}\right) = F\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ \lambda y - \lambda x \end{pmatrix} = \lambda\begin{pmatrix} 2x \\ y-x \end{pmatrix}$
$= \lambda F\begin{pmatrix} x \\ y \end{pmatrix}$. In addition

$\operatorname{Ker}(F) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$ since $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2x = 0 \\ y - x = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$.

$\operatorname{Ran}(F) = \mathbb{R}^2$ since for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, one has $\begin{pmatrix} x/2 \\ y + x/2 \end{pmatrix} \in \mathbb{R}^2$ and $F\begin{pmatrix} x/2 \\ y+x/2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

e) Linearity : $F\left(\binom{x}{y}+\binom{x'}{y'}\right) = F\binom{x+x'}{y+y'} = \binom{y+y'}{x+x'} = \binom{y}{x}+\binom{y'}{x'}$
$= F\binom{x}{y} + F\binom{x'}{y'}$, and $F\left(\lambda\binom{x}{y}\right) = F\binom{\lambda x}{\lambda y} = \binom{\lambda y}{\lambda x} = \lambda\binom{y}{x}$
$= \lambda F\binom{x}{y}$.

In addition $\operatorname{Ker}(F) = \binom{0}{0}$ since $F\binom{x}{y} = \binom{0}{0} \iff \binom{y}{x} = \binom{0}{0}$, and $\operatorname{Ran}(F) = \mathbb{R}^2$ since for any $\binom{x}{y} \in \mathbb{R}^2$, one has $\binom{y}{x} \in \mathbb{R}^2$ and $F\binom{y}{x} = \binom{x}{y}$.

f) Linearity : F is not linear since $F\binom{\lambda x}{\lambda y} = \lambda^2 xy$ but $\lambda F\binom{x}{y} = \lambda xy \neq \lambda^2 xy$ except if $\lambda = 1$.

One has $\operatorname{Ker}(F) = \left\{\binom{x}{y} \in \mathbb{R}^2 \mid xy = 0\right\}$
$= \left\{\binom{0}{y} \in \mathbb{R}^2 \mid y \text{ is arbitrary}\right\} \cup \left\{\binom{x}{0} \in \mathbb{R}^2 \mid x \text{ arbitrary}\right\}$

$\operatorname{Ran}(F) = \mathbb{R}$ since for any $x \in \mathbb{R}$, $\binom{x}{1} \in \mathbb{R}^2$ and $F\binom{x}{1} = x$.

Exercise 3

Let us set $D := \{{}^t(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + x_2 + \dots + x_n = 0\}$.

D is a subspace of $\mathbb{R}^n$. Indeed, if $X = {}^t(x_1, \dots, x_n) \in D$ and $Y = {}^t(y_1, \dots, y_n) \in D$, it means that $x_1 + \dots + x_n = 0$ and $y_1 + \dots + y_n = 0$. As a consequence $X + Y = {}^t(x_1+y_1, \dots, x_n+y_n)$ satisfies $(x_1+y_1) + \dots + (x_n+y_n) = x_1 + \dots + x_n + y_1 + \dots + y_n = 0 + 0 = 0$ and thus $X + Y \in D$.

Similarly $\lambda X \in D$ since $(\lambda x_1) + (\lambda x_2) + \dots + (\lambda x_n) = \lambda(x_1 + x_2 + \dots + x_n)$
$= \lambda\, 0 = 0$.

D is a subspace of dimension $n-1$ since

$V_1 = {}^t(1, 0, 0 \dots 0, -1)$, $V_2 = {}^t(0, 1, 0, \dots, -1)$, $V_j = {}^t(0, 0 \dots, \underset{\text{position } j}{1}, 0 \dots, -1)$

$V_{n-1} = {}^t(0, 0, \dots 0, 1, -1)$ all belong to D and are linearly independent. One can not expect D to be of dimension n since $\begin{pmatrix}1\\0\\0\end{pmatrix} \notin D$.

Exercise 4

1) D is linear since for any $f, g \in C^\infty(\mathbb{R})$ one has

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$, and

$D(\lambda f) = (\lambda f)' = \lambda f' = \lambda D(f)$.

2) One looks for $f \in C^\infty(\mathbb{R})$ such that $D(f) = f' = 0$

But $f' = 0$ if and only if $f = $ cte. Since a constant function admits derivative of all orders, one concludes that $\mathrm{Ker}(D) = \{ \text{constant functions} \} \equiv \mathbb{R}$.

3) $\mathrm{Ker}(D^n) = \{ f \in C^\infty(\mathbb{R}) \mid \underbrace{((f')')' \ldots)'}_{n \text{ times}} = 0 \}$

$= \{ f \mid f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1} \}$

$= \{ \text{polynomial function of degree } n-1 \}$.

This space is of dimension n, and a basis is provided by the functions $f_0(x) = 1$ (constant function), $f_1(x) = x$, $f_2(x) = x^2, \ldots, f_{n-1}(x) = x^{n-1}$.

Exercise 5

a) $\Leftrightarrow \begin{cases} 2x+y=0 \\ z=0 \end{cases} \Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y=-2x \\ z=0 \end{cases}$.

The set of solutions is $\left\{ x\begin{pmatrix}1\\-2\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$ and is thus of dimension 1. A basis is given by $\left\{ \begin{pmatrix}1\\-2\\0\end{pmatrix} \right\}$.

b) $\Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y \text{ arbitrary} \\ z=-x+y \end{cases}$. The set of solutions is

$$\left\{ \begin{pmatrix}x\\y\\-x+y\end{pmatrix} \,\middle|\, x,y \in \mathbb{R} \right\} = \left\{ x\begin{pmatrix}1\\0\\-1\end{pmatrix} + y\begin{pmatrix}0\\1\\1\end{pmatrix} \,\middle|\, x,y \in \mathbb{R} \right\}$$

which is of dimension 2. A basis is given by $\left\{ \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$.

c) $\Leftrightarrow \begin{cases} 9y=(\pi+2)z \\ 2x=\dfrac{\pi-7}{9}z \end{cases} \Leftrightarrow \begin{cases} x=\dfrac{\pi-7}{18}z \\ y=\dfrac{\pi+2}{9}z \\ z \text{ arbitrary} \end{cases}$.

The set of solutions is $\left\{ z\begin{pmatrix}(\pi-7)/18\\(\pi+2)/9\\1\end{pmatrix} \,\middle|\, z \in \mathbb{R} \right\}$, of dimension 1 and a basis is provided by $\left\{ \begin{pmatrix}\pi-7\\2\pi+4\\18\end{pmatrix} \right\}$.

d) $\Leftrightarrow \begin{cases} x=y \\ z=-y \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$. Thus the only solution is $\begin{pmatrix}0\\0\\0\end{pmatrix}$ which is a vector space of dimension 0.