Exercise 1

1) Let $\lambda \in \mathbb{R}$ be an eigenvalue of L_A, i.e. $AX = \lambda X$ for some $X \in \mathbb{R}^n$, $X \neq 0$. Then

$$\langle AX, X\rangle = \langle \lambda X, X\rangle = \lambda \langle X, X\rangle = \lambda \|X\|^2.$$

Since $\langle AY, Y\rangle > 0$ for any $Y \in \mathbb{R}^n$ with $Y \neq 0$ (the assumption), it follows in particular that

$\lambda \|X\|^2 = \langle AX, X\rangle > 0$; and thus $\lambda > 0$.

2) Since A is symmetric ($\Leftrightarrow A = {}^tA$), it follows that

${}^t(A^2) = {}^t(AA) = {}^tA\,{}^tA = AA = A^2 \Rightarrow A^2$ symmetric.

In addition, for any $Y \in \mathbb{R}^n$ one has

$\langle A^2 Y, Y\rangle = \langle AY, AY\rangle = \|AY\|^2 > 0$. Note that $AY \neq 0$ since otherwise one could not have

$\langle AY, Y\rangle > 0 \quad \forall Y \in \mathbb{R}^n$ with $Y \neq 0$.

3) Since $({}^tA)^{-1} = {}^t(A^{-1})$, cf linear algebra I, it follows that ${}^t(A^{-1}) = ({}^tA)^{-1} = A^{-1}$, and thus A^{-1} is symmetric.

Since A is invertible, it follows that $\mathrm{Ran}(L_A) = \mathbb{R}^n$. Thus, for any $Y \in \mathbb{R}^n$, $\exists X \in \mathbb{R}^n$ such that $Y = AX$ (in fact, $X = A^{-1}Y$), and then

$$\begin{aligned}\langle A^{-1}Y, Y\rangle &= \langle A^{-1}AX, AX\rangle = \langle X, AX\rangle \\ &= \langle AX, X\rangle > 0 \text{ by assumption.}\end{aligned}$$

Thus A^{-1} is positive definite.

Exercise 2

Since $\mathcal{A}$ is symmetric, $L_{\mathcal{A}}$ has n eigenvalues, and there exists C invertible such that
$$\mathcal{A} = C^{-1}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} C$$
with λ_j the eigenvalues of $L_{\mathcal{A}}$.

Set $\mathcal{B} = C^{-1}\begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix} C$ which is possible no matter if $\lambda_j \geq 0$ or $\lambda_j < 0$.

Then
$$\mathcal{B}^3 = C^{-1}\begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix} C C^{-1}\begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix} C C^{-1}\begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix} C$$
$$= C^{-1}\begin{pmatrix} \sqrt[3]{\lambda_1} & & \\ & \ddots & \\ & & \sqrt[3]{\lambda_n} \end{pmatrix}^3 C = C^{-1}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} C = \mathcal{A}.$$

Exercise 3

We know frame homework 10, Ex 6. that
$$\begin{pmatrix} 1/5 & 2/5 \\ 4/5 & 3/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \frac{1}{5} B^{-1}\begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix} B$$
$$= B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1/5 \end{pmatrix} B,$$
with B given in .

Then
$$\mathcal{A}^2 = B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1/5 \end{pmatrix} B B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1/5 \end{pmatrix} B = B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1/25 \end{pmatrix} B.$$

Similarly
$$\mathcal{A}^3 = B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1/125 \end{pmatrix} B$$
$$\mathcal{A}^{25} = B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & (-1/5)^{25} \end{pmatrix} B$$
$$\mathcal{A}^{\infty} = B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B.$$

These expressions can be easily computed.

Exercise 4

Observe first that $A = A^*$ which that the spectrum of A is real. Indeed one has

$$P_A(\lambda) = \mathrm{Det}(A - \lambda) = \mathrm{Det}\begin{pmatrix} -\lambda & i \\ -i & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

The eigenvalues of L_A are -1 and 1.

For the eigenspace one has

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} x + iy = 0 \\ -ix + y = 0 \end{cases} \Longleftrightarrow y = ix .$$

Thus, the eigenspace associated with the eigenvalue -1 is $\left\{ x\begin{pmatrix} 1 \\ i \end{pmatrix} \mid x \in \mathbb{C} \right\}$.

Similarly $\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow x = iy$, and the eigenspace associated with the eigenvalue 1 is

$$\left\{ y\begin{pmatrix} i \\ 1 \end{pmatrix} \mid y \in \mathbb{C} \right\}.$$

Observe that the two vectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$ are orthogonal in $\mathbb{C}^2$, indeed $\left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} \right\rangle =$

$$= 1\,\bar{i} + i\,\bar{1} = \bar{i} + i = -i + i = 0.$$

scalar product in $\mathbb{C}^2$.

Exercise 5.

See Homework 10, ex 2.