Homework 11

Exercise 1

⚠ $\mathrm{Re}(z) \equiv \Re(z)$.

No, $\mathrm{Re}(zz') \neq \mathrm{Re}(z)\,\mathrm{Re}(z')$ in general.
For example, if $z = z' = i$, then $\mathrm{Re}(z) = \mathrm{Re}(z') = 0$
but $\mathrm{Re}(zz') = \mathrm{Re}(i^2) = \mathrm{Re}(-1) = -1$.

Exercise 2

i) $(1+3i) + (1-2i) = (1+1) + (3-2)i = 2 + 1i = \underline{2+i}$.

ii) $(2+3i)(1-2i) = 2 - 4i + 3i - 6i^2$

$$= 2 + 6 + (-4+3)i = \underline{8-i}.$$

iii) One looks for $z = x+iy$ such that $(x+iy)i = 1$
$\Leftrightarrow xi - y = 1 \Rightarrow x = 0,\ y = -1$.
Thus, $\underline{i^{-1} = -i}$.

iv) $(1+i)^{-1} = \dfrac{1}{1+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1-i}{2} = \underline{\dfrac{1}{2} - \dfrac{1}{2}i}$

v) One looks for $z = x+iy$, $x, y \in \mathbb{R}$ such that
$(x+iy)^2 = x^2 - y^2 + 2ixy = i$

$$\Leftrightarrow \begin{cases} x^2 - y^2 = 0 \\ 2xy = 1 \end{cases} \Leftrightarrow \begin{cases} y = \pm x \\ \pm x^2 = \frac{1}{2} \end{cases}$$

The solution $-x^2 = \frac{1}{2}$ is not considered since $x \in \mathbb{R}$.

Thus $\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or $\sqrt{i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

There exist 2 square root for i.

Exercise 3

Let $z_1 = x_1 + i y_1 = r_1(\cos\theta_1 + i\sin\theta_2)$ and $z_2 = x_2 + i y_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $|z_1 z_2|^2 = |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|^2$

$= [(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2]$

$= x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2$

$= x_1^2(x_2^2 + y_2^2) + y_1^2(y_2^2 + y_1^2)$

$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2|z_2|^2$

$\Rightarrow \quad \underline{|z_1z_2| = |z_1||z_2|.}$

Alternatively, $z_1z_2 = r_1(\cos(\theta_1) + i\sin(\theta_1))\, r_2(\cos(\theta_2) + i\sin(\theta_2))$

$= r_1r_2[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2)]$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

Since $|\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)| = 1$, one gets

$|z_1z_2| = r_1r_2 = |z_1||z_2|$, and

$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$.

Exercise 4

Since $z^n = z z z \dots z$, one deduces from the previous exercise that if $z = r(\cos(\theta) + i\sin(\theta))$, then

$z^n = r r r \dots r\,[\cos(\theta + \theta + \theta \dots + \theta) + i\sin(\theta + \theta + \theta + \dots + \theta)]$

$= r^n[\cos(n\theta) + i\sin(n\theta)]$.

Exercise 5

Since $\cos(\theta + 2\pi j) = \cos(\theta)$ and $\sin(\theta + 2\pi j) = \sin(\theta)$ for any $j \in \{0, 1, \dots, n-1\}$, one deduces from the previous exercise that if $z = r(\cos(\theta) + i\sin(\theta))$, then

$$z_j := \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi j}{n}\right) + i\sin\left(\frac{\theta + 2\pi j}{n}\right)\right]$$

satisfies $z_j^n = z$, for any $j = 0, 1, 2, \dots, n-1$.

Observe then that if $j \neq k$ and $j, k \in \{0, 1, \dots n-1\}$, then

$$\frac{\theta + 2\pi j}{n} \neq \frac{\theta + 2\pi k}{n} \Rightarrow z_j \neq z_k.$$

Thus, any $z \in \mathbb{C}$ admits n different n-th roots.

Exercise 6

Let $z = x + iy$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

1) $\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2)$
$= \overline{z_1} + \overline{z_2}$.

2) $\overline{z_1 z_2} = \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1)$
$= (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1}\,\overline{z_2}$.

3) $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$.

4) $z^{-1} = \dfrac{1}{x + iy} = \dfrac{1(x - iy)}{(x + iy)(x - iy)} = \dfrac{x - iy}{x^2 + y^2} = \dfrac{\bar{z}}{|z|^2}$.

5) $\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(x + iy + x - iy) = x = \mathrm{Re}(z) = \Re(z)$

$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}\left((x + iy) - (x + iy)\right) = \frac{1}{2i} 2iy = y = \mathrm{Im}(z) = \Im(z)$.

Exercise 7 Let $z = x + iy = r[\cos(\theta) + i\sin(\theta)]$

$$|\bar{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2+y^2} = |z|.$$

$$\bar{z} = x - iy = \sqrt{x^2+y^2}\left[\frac{x}{\sqrt{x^2+y^2}} + i\frac{-y}{\sqrt{x^2+y^2}}\right]$$

$$= r[\cos(\theta) - i\sin(\theta)]$$

$$= r[\cos(-\theta) + i\sin(-\theta)]$$

$$\Rightarrow \arg(\bar{z}) = -\theta = -\arg(z).$$

Exercise 8 $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

1) $$e^{z_1} e^{z_2} = e^{x_1 + iy_1} e^{x_2 + iy_2}$$

$$= e^{x_1}(\cos(y_1) + i\sin(y_1))\, e^{x_2}(\cos(y_2) + i\sin(y_2))$$

$$= e^{x_1} e^{x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$

$$= e^{x_1+x_2 + i(y_1+y_2)}$$

$$= e^{z_1+z_2}$$

$$\Rightarrow e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

2) Since $e^x \neq 0$ for any $x \in \mathbb{R}$ and since $(\cos(y) + i\sin(y)) \neq 0$ $\forall y \in \mathbb{R}$, it follows that $e^z = e^x(\cos(y) + i\sin(y)) \neq 0$ $\forall x, y \in \mathbb{R}$.

3) Since $|\cos(y) + i\sin(y)| = 1$, one has

$$|e^z| = |e^x(\cos(y) + i\sin(y))| = |e^x| = e^x \quad \text{since } e^x > 0$$

4) $$e^{i\pi} = e^0(\cos(\pi) + i\sin(\pi)) = 1(-1+0) = -1.$$