Homework 10

Exercise 1

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$. Then

$$P_A(\lambda) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - cb =$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - \lambda \, Tr(A) + Det(A).$$

Exercise 2

i) Recall that $P_A(\lambda) = Det(A - \lambda 1\!\!1_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$ if A has n eigenvalues. Then, by choosing $\lambda = 0$ one gets $P_A(0) = Det(A - 0) = Det(A) = (\lambda_1 - 0)(\lambda_2 - 0) \ldots (\lambda_n - 0)$
$= \lambda_1 \lambda_2 \ldots \lambda_n$.

ii) Again from $P_A(\lambda)$ one gets
$$P_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda) =$$
$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \ldots + \lambda_n)(-\lambda)^{n-1}$ + polynomial of degree $n-2$.

On the other hand, by computing directly:
$$P_A(\lambda) = Det(A - \lambda 1\!\!1_n) = \sum_{i=1}^{n} (A - \lambda 1\!\!1_n)_{i1} (-1)^{i+1} Det\big((A - \lambda 1\!\!1_n)(i,1)\big)$$
$= (a_{11} - \lambda) \, Det\big((A - \lambda 1\!\!1_n)(1,1)\big)$ + polynomial of degree $n-2$,
by repeating the argument, one obtains
$= (a_{11} - \lambda)(a_{22} - \lambda) \ldots (a_{nn} - \lambda)$ + polynomial of degree $n-2$
$= (-\lambda)^n + (a_{11} + a_{22} + \ldots a_{nn})(-\lambda)^{n-1}$ + polynomial of " ".

Thus, by identifying the coefficients in front of $(-\lambda)^{n-1}$, one obtains $Tr(A) = a_{11} + a_{22} + \ldots + a_{nn}$
$= \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Exercise 3

Recall that if $\mathcal{B} \in M_n(\mathbb{R})$, then $\mathrm{Det}(\mathcal{B}) = \mathrm{Det}({}^t\mathcal{B})$.
Therefore, $P_{{}^t\mathcal{A}}(\lambda) = \mathrm{Det}({}^t\mathcal{A} - \lambda \mathbb{1}_n) = \mathrm{Det}[{}^t(\mathcal{A} - \lambda \mathbb{1}_n)]$
$= \mathrm{Det}(\mathcal{A} - \lambda \mathbb{1}_n) = P_{\mathcal{A}}(\lambda)$. Since $P_{{}^t\mathcal{A}}(\cdot) = P_{\mathcal{A}}(\cdot)$, these functions vanish for the same values of λ, and thus $L_{\mathcal{A}}$ and $L_{{}^t\mathcal{A}}$ have the same eigenvalues.

Exercise 4

Assume that $X \in \mathbb{R}^n$, $X \neq 0$, satisfies $\mathcal{A}X = \lambda X$ for some $\lambda \in \mathbb{R}$. Then one has

$$|\lambda|^2 \|X\|^2 = \|\lambda X\|^2 = \|\mathcal{A}X\|^2 = \langle \mathcal{A}X, \mathcal{A}X \rangle$$
$$= \langle {}^T\!\mathcal{A}\mathcal{A}X, X \rangle = \langle \mathcal{A}^{-1}\mathcal{A}X, X \rangle = \langle X, X \rangle$$
$$= \|X\|^2 .$$

Since $\|X\| \neq 0$, it follows that $|\lambda| = 1$, or that $\lambda = \pm 1$.

Exercise 5

One has $P_A(\lambda) = \det \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{pmatrix} =$

$= (2-\lambda)\left[(1-\lambda)(4-\lambda) + 2 \right] = (2-\lambda)(\lambda^2 - 5\lambda + 6)$
$= (2-\lambda)^2(3-\lambda)$. Thus, the eigenvalues are 2 and 3.

Eigenspace for $\lambda = 2$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y = 0 \\ z = 0 \end{cases}$. The corresponding eigenspace is

$\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$ which is of dimension 1.

Eigenspace for $\lambda = 3$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{cases} x = y \\ 2y = -z \\ 2y = -z \end{cases} \Leftrightarrow \begin{cases} x = y \\ y \text{ arbitrary} \\ z = -2y \end{cases}$.

The corresponding eigenspace is $\left\{ y\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\}$ which is of dimension 1.

Exercise 6

The eigenvalues of L_A are -1 and 5, with corresponding eigenspace $\left\{ x\begin{pmatrix}1\\-1\end{pmatrix} \mid x \in \mathbb{R} \right\}$ and $\left\{ x\begin{pmatrix}1\\2\end{pmatrix} \mid x \in \mathbb{R} \right\}$, respectively.

Now, observe that $B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$,

eigenvector for $\lambda = -1$

eigenvector for $\lambda = 5$

$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, and that

$B^{-1} A B = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, B realizes a change of bases in which the linear map $L_{B^{-1}AB}$ is diagonal.

Exercise 7

$$P_A(\lambda) = \det\begin{pmatrix} -2-\lambda & -7 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 + 3 .$$

Thus, one looks for λ_1 and λ_2 satisfying $\lambda_1^2 = -3$ and $\lambda_2^2 = -3$. Then, one has for $j = 1, 2$:

$$\begin{pmatrix} -2-\lambda_j & -7 \\ 1 & 2-\lambda_j \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \left\{ \begin{array}{l} -(2+\lambda_j)\,x = 7y \\ x = (-2+\lambda_j)\,y \end{array} \right. \quad \begin{array}{l} \cdot 1 \\ \cdot (2+\lambda_j) \end{array}$$

$$\iff \left\{ \begin{array}{l} x = \dfrac{-7}{2+\lambda_j}\, y = \dfrac{-7}{2+\lambda_j} \cdot \dfrac{-2+\lambda_j}{-2+\lambda_j}\, y = \dfrac{-7(-2+\lambda_j)}{\lambda_j^2 - 4}\, y = \dfrac{-7(-2+\lambda_j)}{-7}\, y \\ x = (-2+\lambda_j)\,y \end{array} \right. .$$

Thus, the eigenspaces are $\left\{ y\begin{pmatrix} -2+\lambda_j \\ 1 \end{pmatrix} \mid y \in \mathbb{R} \right\}$ which are of dimension 1.