
Homework 1

Exercise 1 Determine which of the following maps are linear:

- a) $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$,
- b) $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ defined by $F(X) = -X$ for all $X \in \mathbb{R}^4$,
- c) $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $F(X) = X + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ for all $X \in \mathbb{R}^3$,
- d) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y-x \end{pmatrix}$,
- e) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$,
- f) $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $F \begin{pmatrix} x \\ y \end{pmatrix} = xy$.

Exercise 2 Determine the kernel and the range of the maps defined in the previous exercise.

Exercise 3 Consider the subset of \mathbb{R}^n consisting of all vectors ${}^t(x_1, \dots, x_n)$ such that $x_1 + x_2 + \dots + x_n = 0$. Is it a subspace of \mathbb{R}^n ? If so, what is its dimension?

Exercise 4 Let $C^\infty(\mathbb{R})$ be the vector space of all real functions on \mathbb{R} which admit derivatives of all orders. Let $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ be the map which associates to any $f \in C^\infty(\mathbb{R})$ its derivative, i.e. $Df = f'$.

- 1. Is D a linear map?
- 2. What is the kernel of D ?
- 3. What is the kernel of D^n , for any $n \in \mathbb{N}$, and what is the dimension of this vector space?

Exercise 5 What is the dimension of the space of solutions of the following systems of linear equations? In each case, find a basis for the space of solutions.

$$a) \begin{cases} 2x + y - z = 0 \\ 2x + y + z = 0 \end{cases} \quad b) \begin{cases} x - y + z = 0 \end{cases} \quad c) \begin{cases} 4x + 7y - \pi z = 0 \\ 2x - y + z = 0 \end{cases}$$

and

$$d) \begin{cases} x + y + z = 0 \\ x - y = 0 \\ y + z = 0 \end{cases}$$