32150143
Kimura Seigo.
m15021b@math.nagoya-u.ac.jp

K_1-groups and determinants

For each unital C^*-algebra A there is a group homomorphism ω making the diagram

$$\begin{array}{ccc} \mathcal{U}(A) & \xrightarrow{[\cdot]_1} & \\ \downarrow & \searrow & \\ \mathcal{U}(A)/\mathcal{U}_0(A) & \xrightarrow[\omega]{} & K_1(A) \end{array}$$

commutative. That ω exists follows from the fact that $[u]_1 = 0$ for each $u \in \mathcal{U}_0(A)$. ($u \sim_h 1$ for each $u \in \mathcal{U}_0(A)$ and $[1]_1 = 0$)

The map ω is in general neither injective nor surjective, but some important cases ω is actually an isomorphism. In this report we shall investigate the map ω for unital Abelian C^*-algebras.

Example. $A = \mathbb{C}$ then $\mathcal{U}(A) = \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.
$\mathcal{U}_0(A) = \mathcal{U}(A) = \mathbb{T}$
$\mathcal{U}(A)/\mathcal{U}_0(A) = \{1\}$.

For all $u \in \mathcal{U}(\mathbb{C})$ $[u]_1 = [1]_1 = 0$. Let $\langle u \rangle$ denote the equivalence class in $\mathcal{U}(A)/\mathcal{U}_0(A)$ of u in $\mathcal{U}(A)$ and let ω be $\omega(\langle u \rangle) = [u]_1$ for each $\langle u \rangle \in \mathcal{U}(A)/\mathcal{U}_0(A)$, then $\omega(\langle u \rangle) = \omega(1) = [1]_1 = 0$. above diagram commutative.

Let A be an Abelian C^*-algebra. For each natural number n define the determinant $D: M_n(A) \to A$ by

$$D\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{j=1}^{n} a_{j\sigma(j)}$$

where S_n is the group of all permutations of $\{1, 2, \cdots, n\}$. If $A = \mathbb{C}$, then D is the usual determinant.

As with the usual determinant (when $A = \mathbb{C}$) the determinant D has the following properties:

- $D(ab) = D(a)D(b)$ for all a, b in $M_n(A)$
- $D\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = D(a)D(b)$ for all a in $M_n(A)$ and b in $M_m(A)$,
- $D(a^*) = D(a)^*$ for all a in $M_n(A)$
- $D(a) = a$ for all a in A
- $D: M_n(A) \to A$ is continuous for every n.

Thus if A is unital (and still Abelian), then D maps $\mathcal{U}_\infty(A)$ into $\mathcal{U}(A)$

$$\left(\begin{array}{l} \text{Let } a \in \mathcal{U}_\infty(A) \quad \therefore\ a^*a = aa^* = 1_n \quad (a \in \mathcal{U}_n(A)). \\ D(a)^* D(a) = D(a^*)D(a) = D(a^*a) = D(1_n) = 1 \\ \text{and } D(a)D(a)^* = D(a)D(a^*) = D(aa^*) = D(1_n) = 1 \end{array} \right)$$

And
- $D\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = D(u)D(v)$ for u, v in $\mathcal{U}_\infty(A)$
- if u, v belong to $\mathcal{U}_n(A)$ and $u \sim_h v$, then $D(u) \sim_h D(v)$ (Because D is continuous).

From the universal property of K_1 applied to the map.

$$\nu : \mathcal{U}_\infty(A) \to \mathcal{U}(A)/\mathcal{U}_0(A) \quad , \quad \nu(u) = \langle D(u) \rangle$$

We get a homomorphism $\Delta : K_1(A) \to \mathcal{U}(A)/\mathcal{U}_0(A)$ with $\Delta([u]_1) = \langle D(u) \rangle$ for each u in $\mathcal{U}_\infty(A)$.
In this way we get a commutative diagram.

$$\begin{tikzcd}
 & \mathcal{U}(A) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{U}(A)/\mathcal{U}_0(A) \arrow[r, "\omega"'] & K_1(A) \arrow[r, "\Delta"'] & \mathcal{U}(A)/\mathcal{U}_0(A)
\end{tikzcd}$$

In particular $\Delta \circ \omega(\langle u \rangle) = \langle u \rangle$ or $\Delta \circ \omega$ is the identity map on $\mathcal{U}(A)/\mathcal{U}_0(A)$.
We have thus proved the following proposition.

Proposition 8.3.1 Let A be a unital Abelian C^*-algebra. Then

$$(8.6) \qquad 0 \to \mathrm{Ker}(\Delta) \xrightarrow{\iota} K_1(A) \underset{\omega}{\overset{\Delta}{\rightleftarrows}} \mathcal{U}(A)/\mathcal{U}_0(A) \to 0.$$

is a split exact sequence of Abelian groups.
In particular $\omega : \mathcal{U}(A)/\mathcal{U}_0(A) \to K_1(A)$ is injective, and.

$$K_1(A) \cong \mathcal{U}(A)/\mathcal{U}_0(A) \oplus \mathrm{Ker}(\Delta)$$

Note the immediate consequence of Proposition 8.3.1 that $K_1(A)$ is non-zero if A is a unital Abelian C^*-algebra and $\mathcal{U}(A)$ is not connected.
($\mathcal{U}(A)$ is not connected. $\Rightarrow$ There exist some $u_0 \in \mathcal{U}(A)$ that is not homotopic to 1_A. $\Rightarrow$ $\mathcal{U}(A)/\mathcal{U}_0(A)$ has $\langle u_0 \rangle \neq 0$)

If $A = C(X)$ for some compact Hausdorff space X, then $\mathcal{U}(A)$ is equal to the set of continuous functions $C(X, \mathbb{T})$. Because $\varphi^*(x)\varphi(x) = 1$ for all $x \in X \Rightarrow |\varphi(x)|^2 = 1$
$\therefore \varphi(x) \in \mathbb{T}$ for all $x \in X$. The group of homotopy equivalence classes in $C(X, \mathbb{T})$ is called the cohomotopy group of X, and it is denoted by $\pi^1(X)$. In other words,

$$\pi^1(X) = \mathcal{U}(C(X))/\mathcal{U}_0(C(X))$$

The short exact sequence in (8.6) can thus be rewritten as.

$$0 \to \mathrm{Ker}(\Delta) \xrightarrow{\iota} K_1(C(X)) \underset{\omega}{\overset{\Delta}{\rightleftarrows}} \pi^1(X) \to 0$$

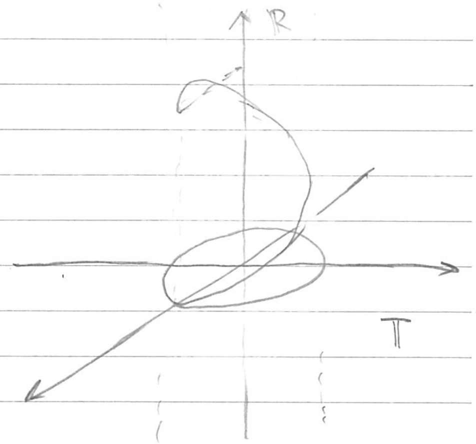
We shall use these facts in the following example.

Example 8.3.2 The cohomotopy group $\pi^1(\mathbb{T})$ is isomorphic to $\mathbb{Z}$. In particular $K_1(C(\mathbb{T})) \neq 0$

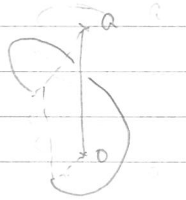
proof For each $u \in C(\mathbb{T}, \mathbb{T})$ there is a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$u(e^{2\pi i t}) = e^{2\pi i f(t)} \qquad t \in [0,1]. \qquad (8.8)$$

If $f, g : [0,1] \to \mathbb{R}$: continuous both satisfy (8.8), then $f - g$ is a constant integer. The map $w : C(\mathbb{T}, \mathbb{T}) \to \mathbb{Z}$ given by $w(u) = f(1) - f(0)$ is therefore well defined. The integer $w(u)$ is called the winding number of u. The winding number map w is surjective,



Because for each $a \in \mathbb{Z}$, we consider following figure.
Then we defines f by

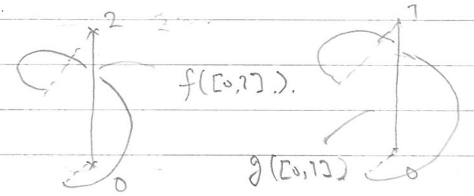


$f : [0,1] \to [0, a]$: continuous
with $f(0) = 0$, $f(1) = a$.

and defines u by

$$u(e^{2\pi i t}) := e^{2\pi i f(t)} \qquad t \in [0,1].$$

Then $u \in C(\mathbb{T}, \mathbb{T})$ and $w(u) = a$.



This case $f - g = 1$

And winding number satisfies

- $u \sim_h v$ if and only if $w(u) = w(v)$
- $w(uv) = w(u) + w(v)$

whenever u, v belong to $C(\mathbb{T}, \mathbb{T})$. Hence $[u] \mapsto w(u)$ defines an isomorphism $\pi^1(\mathbb{T}) \to \mathbb{Z}$, where $[u]$ denotes homotop class in $\pi^1(\mathbb{T})$ of u //.