Chapter 1

C^*-algebras

This chapter is mainly based on the first chapters of the books [Mur90] and [RLL00].

1.1 Basics on C^*-algebras

Definition 1.1.1. A Banach algebra $\mathcal{C}$ *is a complex vector space endowed with an associative multiplication and with a norm* $\|\cdot\|$ *which satisfy for any* $a, b, c \in \mathcal{C}$ *and* $\alpha \in \mathbb{C}$

(i) $(\alpha a)b = \alpha(ab) = a(\alpha b)$,

(ii) $a(b + c) = ab + ac$ *and* $(a + b)c = ac + bc$,

(iii) $\|ab\| \leq \|a\| \, \|b\|$ *(submultiplicativity)*

(iv) $\mathcal{C}$ *is complete with the norm* $\|\cdot\|$.

One says that $\mathcal{C}$ is *Abelian* or *commutative* if $ab = ba$ for all $a, b \in \mathcal{C}$. One also says that $\mathcal{C}$ is *unital* if $\mathbf{1} \in \mathcal{C}$, *i.e.* if there exists an element $\mathbf{1} \in \mathcal{C}$ with $\|\mathbf{1}\| = 1$ such that $\mathbf{1}a = a = a\mathbf{1}$ for all $a \in \mathcal{C}$ [1]. *A subalgebra* $\mathcal{J}$ *of* $\mathcal{C}$ is a vector subspace which is stable for the multiplication. If $\mathcal{J}$ is norm closed, it is a Banach algebra in itself.

Examples 1.1.2. *(i)* $\mathbb{C}$ *or* $M_n(\mathbb{C})$ *(the set of* $n \times n$*-matrices over* $\mathbb{C}$*) are unital Banach algebras.* $\mathbb{C}$ *is Abelian, but* $M_n(\mathbb{C})$ *is not Abelian for any* $n \geq 2$.

(ii) The set $\mathcal{B}(\mathcal{H})$ *of all bounded operators on a Hilbert space* $\mathcal{H}$ *is a unital Banach algebra.*

(iv) The set $\mathcal{K}(\mathcal{H})$ *of all compact operators on a Hilbert space* $\mathcal{H}$ *is a Banach algebra. It is unital if and only if* $\mathcal{H}$ *is finite dimensional.*

[1] Some authors do not assume that $\|\mathbf{1}\| = 1$. It has the advantage that the algebra $\{0\}$ consisting only in the element 0 is unital, which is not the case if one assumes that $\|\mathbf{1}\| = 1$.

(iv) *If Ω is a locally compact topological space, $C_0(\Omega)$ and $C_b(\Omega)$ are Abelian Banach algebras, where $C_b(\Omega)$ denotes the set of all bounded and continuous functions from Ω to $\mathbb{C}$, and $C_0(\Omega)$ denotes the subset of $C_b(\Omega)$ of functions f which* vanish at infinity, i.e. *for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $\sup_{x\in\Omega\setminus K} |f(x)| \leq \varepsilon$. These algebras are endowed with the L^∞-norm, namely $\|f\| = \sup_{x\in\Omega} |f(x)|$. Note that $C_b(\Omega)$ is unital, while $C_0(\Omega)$ is not, except if Ω is compact. In this case, one has $C_0(\Omega) = C(\Omega) = C_b(\Omega)$.*

(v) *If (Ω, μ) is a measure space, then $L^\infty(\Omega)$, the (equivalent classes of) essentially bounded complex functions on Ω is a unital Abelian Banach algebra with the essential supremum norm $\|\cdot\|_\infty$.*

Observe that $\mathbb{C}$ is endowed with the complex conjugation, that $M_n(\mathbb{C})$ is also endowed with an operation consisting of taking the transpose of the matrix, and then the complex conjugate of each entry, and that $C_0(\Omega)$ and $C_b(\Omega)$ are also endowed with the operation consisting in taking the complex conjugate $f \mapsto \overline{f}$. All these additional structures are examples of the following structure:

Definition 1.1.3. *A* C^*-algebra *is a Banach algebra $\mathcal{C}$ together with a map $^* : \mathcal{C} \to \mathcal{C}$ which satisfies for any $a, b \in \mathcal{C}$ and $\alpha \in \mathbb{C}$*

(i) $(a^*)^* = a$,

(ii) $(a + b)^* = a^* + b^*$,

(iii) $(\alpha a)^* = \overline{\alpha} a^*$,

(iv) $(ab)^* = b^* a^*$,

(v) $\|a^* a\| = \|a\|^2$.

The map * is called an involution.

Clearly, if $\mathcal{C}$ is a unital C^*-algebra, then $\mathbf{1}^* = \mathbf{1}$.

Examples 1.1.4. *The Banach algebras described in Examples 1.1.2 are in fact C^*-algebras, once complex conjugation is considered as the involution for complex functions. Note that for $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ the involution consists in taking the adjoint*[2] *of any element $a \in \mathcal{B}(\mathcal{H})$ or $a \in \mathcal{K}(\mathcal{H})$. In addition, let us observe that for a family $\{\mathcal{C}_i\}_{i\in I}$ of C^*-algebras, the direct sum $\oplus_{i\in I}\mathcal{C}_i$, with the pointwise multiplication and involution, and the supremum norm, is also a C^*-algebra.*

Definition 1.1.5. A $*$-homomorphism *φ between two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$ is a linear map $\varphi : \mathcal{C} \to \mathcal{Q}$ which satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for all $a, b \in \mathcal{C}$. If $\mathcal{C}$ and $\mathcal{Q}$ are unital and if $\varphi(\mathbf{1}) = \mathbf{1}$, one says that φ is* unit preserving *or* a unital $*$-homomorphism. *If $\|\varphi(a)\| = \|a\|$ for any $a \in \mathcal{C}$, the $*$-homomorphism is* isometric.

[2]If $\mathcal{H}$ is a Hilbert space with scalar product denoted by $\langle\cdot,\cdot\rangle$ and if $a \in \mathcal{B}(\mathcal{H})$, then its adjoint a^* is defined by the equality $\langle af, g\rangle = \langle f, a^* g\rangle$ for any $f, g \in \mathcal{H}$. If $a \in \mathcal{K}(\mathcal{H})$, then $a^* \in \mathcal{K}(\mathcal{H})$ as well.

A C^*-*subalgebra* of a C^*-algebra $\mathcal{C}$ is a norm closed (non-empty) subalgebra of $\mathcal{C}$ which is stable for the involution. It is clearly a C^*-algebra in itself. In particular, if F is a subset of a C^*-algebra $\mathcal{C}$, we denote by $C^*(F)$ the smallest C^*-subalgebra of $\mathcal{C}$ that contains F. It corresponds to the intersection of all C^*-subalgebras of $\mathcal{C}$ that contains F.

Exercise 1.1.6. *(i) Show that a $*$-homomorphism φ between C^*-algebras is isometric if and only if φ is injective.*

(ii) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism between two C^*-algebras, show that the kernel $\mathsf{Ker}(\varphi)$ *of φ is a C^*-subalgebra of $\mathcal{C}$ and that the image* $\mathsf{Ran}(\varphi)$ *of φ is a C^*-subalgebra of $\mathcal{Q}$.*

An important result about C^*-algebras states that each of them can be represented faithfully in a Hilbert space. More precisely:

Theorem 1.1.7 (Gelfand-Naimark-Segal (GNS) representation)**.** *For any C^*-algebra $\mathcal{C}$ there exists a Hilbert space $\mathcal{H}$ and an injective $*$-homomorphism from $\mathcal{C}$ to $\mathcal{B}(\mathcal{H})$. In other words, every C^*-algebra $\mathcal{C}$ is $*$-isomorphic*[3] *to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.*

Extension 1.1.8. *The proof of this theorem is based on the notion of* states *(positive linear functionals) on a C^*-algebra, and on the existence of sufficiently many such states. The construction is rather explicit and can be studied, see for example [Mur90, Thm. 3.4.1].*

The next definition of an ideal is the most suitable one in the context of C^*-algebra.

Definition 1.1.9. An ideal *in a C^*-algebra $\mathcal{C}$ is a (non-trivial) C^*-subalgebra $\mathcal{J}$ of $\mathcal{C}$ such that $ab \in \mathcal{J}$ and $ba \in \mathcal{J}$ whenever $a \in \mathcal{J}$ and $b \in \mathcal{C}$. This ideal $\mathcal{J}$ is said to be* maximal *in $\mathcal{C}$ if $\mathcal{J}$ is proper ($\Leftrightarrow$ not equal to $\mathcal{C}$) and if $\mathcal{J}$ is not contained in any other proper ideal of $\mathcal{C}$.*

For example, $C_0(\Omega)$ is an ideal of $C_b(\Omega)$, while $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$. Let us add one more important result about the quotient of a C^*-algebra by any of its ideals. In this setting we set

$$\mathcal{C}/\mathcal{J} = \{a + \mathcal{J} \mid a \in \mathcal{C}\} \quad \text{and} \quad \|a + \mathcal{J}\| := \inf_{b \in \mathcal{J}} \|a + b\|.$$

In this way $\mathcal{C}/\mathcal{J}$ becomes a C^*-algebra, and if one sets $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ by $\pi(a) = a + \mathcal{J}$, then π is a $*$-homomorphism with $\mathcal{J} = \mathsf{Ker}(\pi)$. The $*$-homomorphism π is called *the quotient map*. We refer to [Mur90, Thm. 3.1.4] for the proof about the quotient $\mathcal{C}/\mathcal{J}$.

Consider now a (finite or infinite) sequence of C^*-algebras and $*$-homomorphisms

$$\ldots \longrightarrow \mathcal{C}_n \xrightarrow{\varphi_n} \mathcal{C}_{n+1} \xrightarrow{\varphi_{n+1}} \mathcal{C}_{n+2} \longrightarrow \ldots$$

[3] A $*$-isomorphism is a bijective $*$-homomorphism.

This sequence is *exact* if $\mathsf{Ran}(\varphi_n) = \mathsf{Ker}(\varphi_{n+1})$ for any n. A sequence of the form

$$0 \longrightarrow \mathcal{J} \stackrel{\varphi}{\longrightarrow} \mathcal{C} \stackrel{\psi}{\longrightarrow} \mathcal{Q} \longrightarrow 0 \tag{1.1}$$

is called *a short exact sequence*. In particular, if $\mathcal{J}$ is an ideal in $\mathcal{C}$ we can consider

$$0 \longrightarrow \mathcal{J} \stackrel{\iota}{\hookrightarrow} \mathcal{C} \stackrel{\pi}{\longrightarrow} \mathcal{C}/\mathcal{J} \longrightarrow 0$$

where ι is the inclusion map and π the quotient map already introduced.

If in (1.1) there exists a $*$-homomorphism $\lambda : \mathcal{Q} \to \mathcal{C}$ such that $\psi \circ \lambda = \mathrm{id}$, then λ is called *a lift for* ψ, and the short exact sequence is said to be *split exact*. For example, let $\mathcal{C}_1$, $\mathcal{C}_2$ be C^*-algebras, and consider the direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ with the pointwise multiplication and involution, and the supremum norm. One can then observe that the following short exact sequence

$$0 \longrightarrow \mathcal{C}_1 \stackrel{\iota_1}{\longrightarrow} \mathcal{C}_1 \oplus \mathcal{C}_2 \stackrel{\pi_2}{\longrightarrow} \mathcal{C}_2 \longrightarrow 0$$

is split exact, when ι_1 and π_2 are defined by $\iota_1(a) = (a, 0)$ and $\pi_2(a, b) = b$. Indeed, one can set $\lambda : \mathcal{C}_2 \to \mathcal{C}_1 \oplus \mathcal{C}_2$ with $\lambda(b) = (0, b)$ and the equality $\pi_2 \circ \lambda = \mathrm{id}$ holds. Note that neither all short exact sequences are split exact, nor all split exact short exact sequences are direct sums.

Let us finally mention that with any C^*-algebra $\mathcal{C}$ one can associate a unique unital C^*-algebra $\widetilde{\mathcal{C}}$ which contains $\mathcal{C}$ as an ideal and such that $\widetilde{\mathcal{C}}/\mathcal{C} = \mathbb{C}$. In addition, the short exact sequence

$$0 \longrightarrow \mathcal{C} \stackrel{\iota}{\hookrightarrow} \widetilde{\mathcal{C}} \stackrel{\pi}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

is split exact, with $\lambda(\alpha) = \alpha\mathbf{1}$ for any $\alpha \in \mathbb{C}$. Here $\mathbf{1}$ denotes the identity element of $\widetilde{\mathcal{C}}$. The C^*-algebra $\widetilde{\mathcal{C}}$ is called *the (smallest) unitization of* $\mathcal{C}$. Note that

$$\widetilde{\mathcal{C}} = \big\{a + \alpha\mathbf{1} \mid a \in \mathcal{C}, \alpha \in \mathcal{C}\big\}, \tag{1.2}$$

and therefore $\mathcal{C}$ is naturally identified with the element of the form $a + 0\mathbf{1}$ in $\widetilde{\mathcal{C}}$.

Exercise 1.1.10. *Work out the details of the construction of* $\widetilde{\mathcal{C}}$*, see for example [RLL00, Exercise 1.3].*

An important property of the previous construction is its functoriality, in the sense that for any $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{Q}$ between C^*-algebras, there exists a unique unit preserving $*$-homomorphism $\tilde{\varphi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{Q}}$ such that $\tilde{\varphi} \circ \iota_{\mathcal{C}} = \iota_{\mathcal{Q}} \circ \varphi$. This morphism is defined by $\tilde{\varphi}(a + \alpha\mathbf{1}_{\widetilde{\mathcal{C}}}) = \varphi(a) + \alpha\mathbf{1}_{\widetilde{\mathcal{Q}}}$ for any $a \in \mathcal{C}$ and $\alpha \in \mathbb{C}$.

1.2 Spectral theory

Let us now consider an arbitrary unital C^*-algebra $\mathcal{C}$, and let $a \in \mathcal{C}$. One says that a is *invertible* if there exists $b \in \mathcal{C}$ such that $ab = \mathbf{1} = ba$. In this case, the element b is denoted by a^{-1} and is called *the inverse of* a. The set of all invertible elements is denoted by $\mathcal{GL}(\mathcal{C})$. Clearly, $\mathcal{GL}(\mathcal{C})$ is a group.

Exercise 1.2.1. *Show that* $\mathcal{GL}(\mathcal{C})$ *is an open set in any unital* C^**-algebra* $\mathcal{C}$*, and that the map* $\mathcal{GL}(\mathcal{C}) \ni a \mapsto a^{-1} \in \mathcal{C}$ *is differentiable. The Neumann series can be used in the proof, namely if* $\|a\| < 1$ *one has*

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n. \tag{1.3}$$

Note that in the sequel, we shall sometimes write $a - z$ for $a - z\mathbf{1}$, whenever a is an element of a unital C^*-algebra and $z \in \mathbb{C}$.

Definition 1.2.2. *Let* $\mathcal{C}$ *be a unital* C^**-algebra and let* $a \in \mathcal{C}$*.* The spectrum $\sigma_{\mathcal{C}}(a)$ of a with respect to $\mathcal{C}$ *is defined by*

$$\sigma_{\mathcal{C}}(a) := \big\{ z \in \mathbb{C} \mid (a - z\mathbf{1}) \notin \mathcal{GL}(\mathcal{C}) \big\}.$$

The spectral radius $r(a)$ of a with respect to $\mathcal{C}$ *is defined by*

$$r(a) := \sup \big\{ |z| \mid z \in \sigma_{\mathcal{C}}(a) \big\}.$$

Note that the spectrum $\sigma_{\mathcal{C}}(a)$ of a is a closed subset of $\mathbb{C}$ which is never empty. This result is not completely trivial and its proof is based on Liouville's Theorem in complex analysis. In addition, note that the estimate $r(a) \leq \|a\|$ and the equality $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ always hold. We refer to [Mur90, Sec. 1.2] for the proofs of these statements. Let us mention that if $\mathcal{C}$ has no unit, the spectrum of an element $a \in \mathcal{C}$ can still be defined by $\sigma_{\mathcal{C}}(a) := \sigma_{\widetilde{\mathcal{C}}}(a)$.

Based on these observations, we state two results which are often quite useful.

Theorem 1.2.3 (Gelfand-Mazur)**.** *If* $\mathcal{C}$ *is a unital* C^**-algebra in which every non-zero element is invertible, then* $\mathcal{C} = \mathbb{C}\mathbf{1}$*.*

Proof. We know from the observation made above that for any $a \in \mathcal{C}$, there exists $z \in \mathbb{C}$ such that $a - z\mathbf{1} \notin \mathcal{GL}(\mathcal{C})$. By assumption, it follows that $a - z\mathbf{1} = 0$, which means $a = z\mathbf{1}$. □

Lemma 1.2.4. *Let* $\mathcal{J}$ *be a maximal ideal of a unital Abelian* C^**-algebra* $\mathcal{C}$*, then* $\mathcal{C}/\mathcal{J} = \mathbb{C}\mathbf{1}$*.*

Proof. As already mentioned, $\mathcal{C}/\mathcal{J}$ is a C^*-algebra with unit $\mathbf{1} + \mathcal{J}$; we denote the quotient map $\mathcal{C} \to \mathcal{C}/\mathcal{J}$ by π. If $\mathcal{I}$ is an ideal in $\mathcal{C}/\mathcal{J}$, then $\pi^{-1}(\mathcal{I})$ is an ideal of $\mathcal{C}$ containing $\mathcal{J}$, which is therefore either equal to $\mathcal{C}$ or to $\mathcal{J}$, by the maximality of $\mathcal{J}$. Consequently, $\mathcal{I}$ is either equal to $\mathcal{C}/\mathcal{J}$ or to 0, and $\mathcal{C}/\mathcal{J}$ has no proper ideal.

Now, if $a \in \mathcal{C}/\mathcal{J}$ and $a \neq 0$, then $a \in \mathcal{GL}\big(\mathcal{C}/\mathcal{J}\big)$, since otherwise $a(\mathcal{C}/\mathcal{J})$ would be a proper ideal of $\mathcal{C}/\mathcal{J}$. In other words, one has obtained that any non-zero element of $\mathcal{C}/\mathcal{J}$ is invertible, which implies that $\mathcal{C}/\mathcal{J} = \mathbb{C}\mathbf{1}$, by Theorem 1.2.3. □

The following statement is an important result for spectral theory in the framework of C^*-algebras. It shows that the computation of the spectrum does not depend on the surrounding algebra.

Theorem 1.2.5. *Let $\mathcal{C}$ be a C^*-subalgebra of a unital C^*-algebra $\mathcal{Q}$ which contains the unit of $\mathcal{Q}$. Then for any $a \in \mathcal{C}$,*

$$\sigma_{\mathcal{C}}(a) = \sigma_{\mathcal{Q}}(a).$$

The proof of this theorem is mainly based on the previous lemmas, but requires some preliminary works. We refer to [Mur90, Thm. 1.2.8 & 2.1.11] for its proof. Note that because of this result, it is common to denote by $\sigma(a)$ the spectrum of an element a of a C^*-algebra, without specifying in which algebra the spectrum is computed.

In the next definition we consider some special elements of a C^*-algebra.

Definition 1.2.6. *Let $\mathcal{C}$ be a C^*-algebra and let $a \in \mathcal{C}$. The element a is* self-adjoint *or* hermitian *if $a = a^*$, a is* normal *if $aa^* = a^*a$. If a is self-adjoint and $\sigma(a) \subset \mathbb{R}_+$, then a is said to be positive. If $\mathcal{C}$ is unital and if $u \in \mathcal{C}$ satisfies $uu^* = u^*u = \mathbf{1}$, then u is said to be* unitary.

The set of all positive elements in $\mathcal{C}$ is usually denoted by $\mathcal{C}^+$, and one simply writes $a \geq 0$ to mean that a is positive. An important result in this context is that for any $a \in \mathcal{C}^+$, there exists $b \in \mathcal{C}$ such that $a = b^*b$. One can even strengthen this result by showing that for any $a \in \mathcal{C}^+$, there exists a unique $b \in \mathcal{C}^+$ such that $a = b^2$. This element b is usually denoted by $a^{1/2}$. Now, for any self-adjoint operators a_1, a_2, one writes $a_1 \geq a_2$ if $a_1 - a_2 \geq 0$. For completeness, we add some information about $\mathcal{C}^+$.

Proposition 1.2.7. *Let $\mathcal{C}$ be a C^*-algebra. Then,*

(i) The sum of two positive elements of $\mathcal{C}$ is a positive element of $\mathcal{C}$,

(ii) The set $\mathcal{C}^+$ is equal to $\{a^*a \mid a \in \mathcal{C}\}$,

(iii) If a, b are self-adjoint elements of $\mathcal{C}$ and if $c \in \mathcal{C}$, then $a \geq b \Rightarrow c^*ac \geq c^*bc$,

(iv) If $a \geq b \geq 0$, then $a^{1/2} \geq b^{1/2}$,

(v) If $a \geq b \geq 0$, then $\|a\| \geq \|b\|$,

(vi) If $\mathcal{C}$ is unital and a, b are positive and invertible elements of $\mathcal{C}$, then $a \geq b \Rightarrow b^{-1} \geq a^{-1} \geq 0$,

(vii) For any $a \in \mathcal{C}$ there exist $a_1, a_2, a_3, a_4 \in \mathcal{C}^+$ such that

$$a = a_1 - a_2 + ia_3 - ia_4.$$

Proof. See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur90]. □

In the next statement, we provide some information on the spectrum of self-adjoint and unitary elements of a unital C^*-algebra. For that purpose, we immediately infer from the equality $\|u^*u\| = \|u\|^2$ that if u is unitary, then $\|u\| = 1$. We also set

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Lemma 1.2.8. *Any self-adjoint element a in a unital C^*-algebra $\mathcal{C}$ satisfies $\sigma(a) \subset \mathbb{R}$. If u is a unitary element of $\mathcal{C}$, then $\sigma(u) \subset \mathbb{T}$.*

Proof. First of all, let $b \in \mathcal{C}$ and observe that from the equality $\big((b-z)^{-1}\big)^* = (b^* - \overline{z})^{-1}$, one infers that if $z \in \sigma(b)$, then $\overline{z} \in \sigma(b^*)$. Furthermore, from the equality

$$z^{-1}(z-b)b^{-1} = -(z^{-1} - b^{-1}),$$

one also deduces that if $z \in \sigma(b)$ for some $b \in \mathcal{GL}(\mathcal{C})$, then $z^{-1} \in \sigma(b^{-1})$.

Now, for a unitary $u \in \mathcal{C}$, one deduces from the above computations that if $z \in \sigma(u)$, then $\overline{z}^{-1} \in \sigma\big((u^*)^{-1}\big) = \sigma(u)$. Since $\|u\| = 1$ one then infers from the equality $r(u) = \|u\| = 1$ that $|z| \leq 1$ and $|z^{-1}| \leq 1$, which means $z \in \mathbb{T}$.

If $a = a^* \in \mathcal{C}$, one sets $e^{ia} := \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$ and observes that

$$(e^{ia})^* = e^{-ia} = (e^{ia})^{-1}.$$

Therefore, e^{ia} is a unitary element of $\mathcal{C}$ and it follows that $\sigma\big(e^{ia}\big) \subset \mathbb{T}$. Now, let us assume that $z \in \sigma(a)$, set $b := \sum_{n=1}^{\infty} \frac{i^n(a-z)^{n-1}}{n!}$, and observe that b commutes with a. Then one has

$$e^{ia} - e^{iz} = (e^{i(a-z)} - 1)e^{iz} = (a-z)be^{iz}.$$

It follows from this equality that $e^{iz} \in \sigma(e^{ia})$. Indeed, if $\big(e^{ia} - e^{iz}\big) \in \mathcal{GL}(\mathcal{C})$, then $be^{iz}\big(e^{ia} - e^{iz}\big)^{-1}$ would be an inverse for $(a-z)$, which can not be since $z \in \sigma(a)$. From the preliminary computation, one deduces that $|e^{iz}| = 1$, which holds if and only if $z \in \mathbb{R}$. One has thus obtains that $\sigma(a) \subset \mathbb{R}$. □

Let us now state an important result for Abelian C^*-algebras.

Theorem 1.2.9 (Gelfand)**.** *Any Abelian C^*-algebra $\mathcal{C}$ is $*$-isomorphic to a C^*-algebra of the form $C_0(\Omega)$ for some locally compact Hausdorff*[4] *space Ω.*

In fact, Gelfand's theorem provides more information, namely

(i) The mentioned $*$-isomorphism is isometric,

(ii) Ω is compact if and only if $\mathcal{C}$ is unital,

(iii) Ω and Ω' are homeomorphic if and only if $C_0(\Omega)$ and $C_0(\Omega')$ are $*$-isomorphic,

(iv) The set Ω is called *the spectrum* of $\mathcal{C}$ and corresponds to the set of *characters* of $\mathcal{C}$ endowed with a suitable topology. A character on $\mathcal{C}$ is a non-zero $*$-homomorphism from $\mathcal{C}$ to $\mathbb{C}$.

[4] A Hausdorff space is a topological space in which distinct points have disjoint neighbourhoods.

In this context, let us mention that there exists a bijective correspondence between open subsets of Ω and ideals in $C_0(X)$. For example, if X is any open subset of Ω, then $C_0(X) \subset C_0(\Omega)$ (by extending the element of $C_0(X)$ by 0 on $\Omega \setminus X$) and $C_0(X)$ is then clearly an ideal of $C_0(\Omega)$. As a consequence, one gets the following short exact sequence:

$$0 \longrightarrow C_0(X) \stackrel{\iota}{\hookrightarrow} C_0(\Omega) \stackrel{\pi}{\longrightarrow} C_0(\Omega \setminus X) \longrightarrow 0.$$

Extension 1.2.10. *Write down the details of the construction of the Gelfand transform, first for Banach algebras, and then for C^*-algebras. Provide a proof of the above statements.*

The Gelfand representation has various useful applications. One is contained in the proof of the following statement, see [Mur90, Thm. 2.1.13] for its proof. This statement corresponds to a so-called *bounded functional calculus.*

Proposition 1.2.11. *Let a be a normal element of a unital C^*-algebra $\mathcal{C}$, and let $\iota : \sigma(a) \to \mathbb{C}$ be the inclusion map,* i.e. *$\iota(z) = z$ for any $z \in \sigma(a)$. Then there exists a unique unital $*$-homomorphism $\varphi_a : C\big(\sigma(a)\big) \to \mathcal{C}$ satisfying $\varphi_a(\iota) = a$. Moreover, φ_a is isometric and the image of φ_a is the C^*-subalgebra $C^*(\{a, \mathbf{1}\})$ of $\mathcal{C}$ generated by a and $\mathbf{1}$.*

Note that if f is a polynomial, then the equality $\varphi_a(f) = f(a)$ holds, and if f corresponds to the map $f(z) = \bar{z}$, then one has $\varphi_a(f) = a^*$. For the former reason, one usually write simply $f(a)$ instead of $\varphi_a(f)$ for any $f \in C\big(\sigma(a)\big)$. We also mention a useful result about the spectrum of elements obtained by the previous bounded functional calculus [Mur90, Thm. 2.1.14].

Theorem 1.2.12 (Spectral mapping theorem)**.** *Let a be a normal element in a unital C^*-algebra $\mathcal{C}$, and let φ_a be the $*$-homomorphism mentioned in the previous statement. Then for any $f \in C\big(\sigma(a)\big)$, the following equality holds:*

$$\sigma\big(f(a)\big) = f\big(\sigma(a)\big).$$

Let us still gather some additional spectral properties.

(i) If $\varphi : \mathcal{C} \to \mathcal{Q}$ is a unital $*$-homomorphism between unital C^*-algebras, and if a is a normal element of $\mathcal{C}$, then $\sigma\big(\varphi(a)\big) \subset \sigma(a)$, or in other words the spectrum of a can not increase through a $*$-homomorphism. In addition, if $f \in C\big(\sigma(a)\big)$, then $f\big(\varphi(a)\big) = \varphi\big(f(a)\big)$.

(ii) If a is a normal element in a non-unital C^*-algebra $\mathcal{C}$, then $f(a)$ is *a priori* defined only in its unitization $\widetilde{\mathcal{C}}$. Now, if $\pi : \widetilde{\mathcal{C}} \to \mathbb{C}$ denotes the quotient map and for $a \in \mathcal{C}$, one has by the previous point that

$$\pi\big(f(a)\big) = f\big(\pi(a)\big) = f(0).$$

It thus follows from the description of $\widetilde{\mathcal{C}}$ provided in (1.2) that $f(a)$ belongs to $\mathcal{C}$ if and only if $f(0) = 0$.

(iii) If a is a normal element in a C^*-algebra, then $r(a) = \|a\|$.

We finally state a technical result which will be used at several occasions in the next chapter.

Lemma 1.2.13. *Let $\mathcal{C}$ be a unital C^*-algebra, let K be a non-empty compact subset of $\mathbb{R}$ and let F_K be the set of self-adjoint elements of $\mathcal{C}$ with spectrum in K. Then for any fixed $f \in C(K)$, the map*

$$F_k \ni a \mapsto f(a) \in \mathcal{C}$$

is continuous.

The proof of this statement is provided in [RLL00, Lem. 1.2.5] and relies on an $\varepsilon/3$-argument.

1.3 Matrix algebras

For any C^*-algebra $\mathcal{C}$, let us denote by $M_n(\mathcal{C})$ the set of all $n \times n$ matrices with entries in $\mathcal{C}$. Addition, multiplication and involution for such matrices are mimicked from the scalar case, *i.e.* when $\mathcal{C} = \mathbb{C}$. In order to define a C^*-norm on $M_n(\mathcal{C})$, let us consider any injective $*$-homomorphism $\varphi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and extend this morphism to a $*$-homomorphism $\varphi : M_n(\mathcal{C}) \to \mathcal{B}(\mathcal{H}^n)$ by defining[5]

$$\varphi \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \varphi(a_{11})f_1 + \cdots + \varphi(a_{1n})f_n \\ \vdots \\ \varphi(a_{n1})f_1 + \cdots + \varphi(a_{nn})f_n \end{pmatrix}$$

for any ${}^t(f_1, \dots, f_n) \in \mathcal{H}^n$ (the notation ${}^t(\dots)$ means the transpose of a vector). Then a C^*-norm on $M_n(\mathcal{C})$ is obtained by setting $\|a\| := \|\varphi(a)\|$ for any $a \in M_n(\mathcal{C})$, and this norm is independent of the choice of φ. Note that the following inequalities hold:

$$\max_{i,j} \|a_{ij}\| \leq \left\| \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \right\| \leq \sum_{i,j} \|a_{ij}\|. \tag{1.4}$$

These inequalities have a useful application. It shows that if Ω is a topological space and if $f : \Omega \to M_n(\mathcal{C})$, then f is continuous if and only if each function $f_{ij} : \Omega \to \mathcal{C}$ is continuous.

[5]The use of the same notation for the maps $\varphi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ and $\varphi : M_n(\mathcal{C}) \to \mathcal{B}(\mathcal{H}^n)$ is done on purpose. Some authors would use φ_n for the second map, but the omission of the index n does not lead to any confusion and simplifies the notation.

Finally, let us mention that if $\varphi : \mathcal{C} \to \mathcal{Q}$ is a $*$-homomorphism between two C^*-algebras $\mathcal{C}$ and $\mathcal{Q}$, then the map $\varphi : M_n(\mathcal{C}) \to M_n(\mathcal{Q})$ defined by

$$\varphi \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & \dots & \varphi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \varphi(a_{n1}) & \dots & \varphi(a_{nn}) \end{pmatrix} \tag{1.5}$$

is a $*$-homomorphism, for any $n \in \mathbb{N}^*$. Note that again we have used the same notation for two related but different maps.