Introduction

These lecture notes correspond to a first course in linear algebra, which does not rely on any prerequisite. All necessary notions are introduced from scratch, and the proofs of most of the statements are provided. Examples are provided in the text while exercises are gathered at the end of each chapter. The main source of inspiration has been the book of S. Lang: *Introduction to linear algebra*[1].

0.1 Motivation

Several problems lead naturally to the basic concepts of linear algebra. We list some examples which are at the root of this course or which provide some motivation for this course.

Solving linear equations Consider the following linear system of equations

$$\left\{ \begin{array}{l} 2x + y = 7 \\ -x + 2y = 4 \end{array} \right. .$$

Its only solution is $\begin{cases} x = 2 \\ y = 3 \end{cases}$. However, if one considers the system

$$\left\{ \begin{array}{l} 2x + y = 7 \\ 4x + 2y = 14 \end{array} \right.$$

then one can find several solutions, as for example $\begin{cases} x = 0 \\ y = 7 \end{cases}$ or $\begin{cases} x = 2 \\ y = 3 \end{cases}$. It is then natural to wonder what are all solutions of this system ? How can one describe this set of solutions and how can one understand it ?

Solutions of linear differential equations Consider a real function f defined on $\mathbb{R}$, *i.e.* $f : \mathbb{R} \ni t \mapsto f(t) \in \mathbb{R}$ satisfying the relation

$$f''(t) = -m^2 f(t)$$

[1]Serge Lang, *Introduction to linear algebra*, second edition, Undergraduate texts in mathematics, Springer.

for some constant $m \in \mathbb{R}$. Can one find all solutions f for this equation ? For example, $f(t) = \cos(mt)$, $f(t) = 3\sin(mt)$ or $f(t) = 2\cos(mt+3)$ are solutions of this equation, but how can one describe all of them ?

Describing linear transformations Consider the following linear transformation:

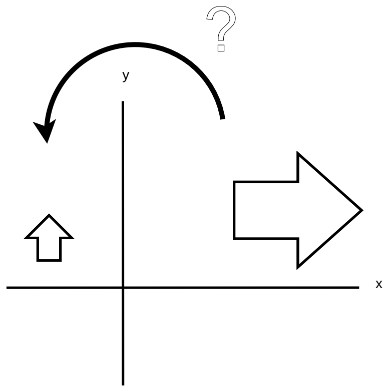

Figure 1: Linear transformation

How can one describe this transformation efficiently ?

Change of bases Consider the same object described in two different reference systems:

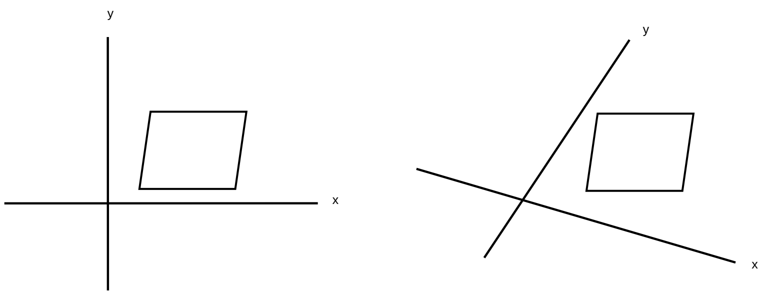

Figure 2: Change of reference system

How can one compare the information on the same object in these two systems ?

Reading a scientific paper How can one understand the following paper: "The \$25'000'000'000 eigenvector: the linear algebra behind Google"[2].

Other motivations Linear algebra is also at the root of quantum mechanics, dynamical systems, linear response theory, linear perturbations theory, ...

[2]Kurt Bryan, Tanya Leise, *The \$25'000'000'000 eigenvector: the linear algebra behind Google*, SIAM REVIEW, Vol. 48, No. 3, pp. 569–581.