Exercises (Report)

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Exercise 1.1.6 Let C, Q : C^*-algebras.

(i) Show that a $*$-homomorphism $\varphi : C \to Q$ is isometric if and only if φ is injective.

Proof:

("only if" part) Since φ : isometry,

$a \neq b \Leftrightarrow \|a-b\| \neq 0 \Longleftrightarrow \|\varphi(a-b)\| \neq 0 \Longleftrightarrow \|\varphi(a)-\varphi(b)\| \neq 0$
$\Longleftrightarrow \varphi(a) \neq \varphi(b)$.

Hence $\varphi : C \to Q$: injective.

("if" part) We may assume that C has an identity.

(For this assumption, see P.8 of lecture note.)

Claim 1: For a self adjoint operator $a \in C$, $\|a\|\mathbb{1} - a \geq 0$.

($\because$) Since $\|a\|\mathbb{1} - a$: self adjoint, it suffices to show

$(\|a\|\mathbb{1} - a) - \lambda\mathbb{1}$ is invertible for any $\lambda < 0$. But since

$\|a\|\mathbb{1} - a - \lambda\mathbb{1} = (\|a\| - \lambda)(\mathbb{1} - \frac{a}{\|a\|-\lambda})$, $\|a\| - \lambda > 0$ and

$\mathbb{1} - \frac{a}{\|a\|-\lambda}$: invertible by the formula of Neumann series,

$\therefore \|a\|\mathbb{1} - a - \lambda\mathbb{1}$: invertible for all $\lambda < 0$. This completes the proof of claim 1. //

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Claim2 A $*$-homomorphism $\varphi : C \to Q$ is norm-decreasing.

($\because$) Let $a \in C$. Then, by "Claim1",

$$\|a^*a\|\mathbb{1} - a^*a \geq 0.$$

Since any $*$-homomorphism φ transforms a positive element to a positive one, we have

$$\|a^*a\| \varphi(\mathbb{1}) \geq \varphi(a)^*\varphi(a) \geq 0.$$

Hence we have

$$\|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| \leq \|a^*a\| \cdot \|\varphi(\mathbb{1})\|.$$

Now, since $\mathbb{1}^2 = \mathbb{1} = \mathbb{1}^*$, $\varphi(\mathbb{1})$ is a projection in Q.

Hence $\|\varphi(\mathbb{1})\| \leq 1$. Therefore we have

$$\|\varphi(a)\| \leq \sqrt{\|a^*a\|} = \|a\|.$$

This completes the proof of Claim2. //

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Claim 3 Let $\mathcal{A}$: commutative C^*-algebra with identity and $\|\cdot\|_1$: another norm under which $(\mathcal{A}, \|\cdot\|_1)$ is a Banach algebra.

Then $\|a\| \leq \|a\|_1$ for any $a \in \mathcal{A}$.

(For this proof, please see "C^*-algebras and W^*-algebras" written by "Shôichiro Sakai".)

(Proof of Exercise 1.1.6 (i))

Let $a \in \mathcal{C}$. By Claim 2, $\|\varphi(a)\| \leq \|a\|$.

Hence it suffices to show $\|\varphi(a)\| \geq \|a\|$.

Let $\mathcal{A}$: commutative C^*-algebra with identity generated by a^*a and $\mathbb{1}$.

Put $\|x\|_1 := \|\varphi(x)\|$ for any $x \in \mathcal{A}$.

Then, since φ : norm-decreasing and injective $*$-homomorphism, $\|\cdot\|_1$ defines a Banach norm on $\mathcal{A}$.

Hence, by claim3, $\|a^*a\| \leq \|a^*a\|_1 = \|\varphi(a^*a)\|$.

Therefore, we have $\|a\|^2 = \|a^*a\| \leq \|\varphi(a)^*\varphi(a)\| = \|\varphi(a)\|^2$.

$\therefore \|a\| = \|\varphi(a)\| \quad (\forall a \in C)$. ▨

(ii) If $\varphi : C \to Q$: $*$-homomorphism, the Kernel $\mathrm{Ker}(\varphi)$ and the image $\mathrm{Ran}(\varphi)$ are C^*-subalgebras of C and Q respectively.

(Proof)

By Claim2 in the proof of (i), φ is norm-decreasing.

Hence φ is continuous. This implies

$\mathrm{Ker}(\varphi) = \varphi^{-1}(\{0\})$: closed ($\because$ $\{0\}$: closed in Q).

Since φ : $*$-homomorphism, $\mathrm{Ker}(\varphi)$: $*$-subalgebra.

Therefore, $\mathrm{Ker}(\varphi)$ is a C^*-subalgebra of C. // ($\mathrm{Ker}(\varphi)$)

Consider the commutative diagram:

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & C/\mathrm{Ker}(\varphi) \arrow[d, "\psi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Q \end{tikzcd}$$

(Where $\pi : C \ni a \mapsto a + \mathrm{Ker}(\varphi) \in C/\mathrm{Ker}(\varphi$

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By the homomorphism theorem, ψ : isomorphism from $\mathcal{C}/\mathrm{Ker}(\varphi)$ to $\mathrm{Ran}(\varphi)$

Moreover, since $\mathrm{Ker}(\varphi)$: (closed) $*$-ideal,

$\psi : \mathcal{C}/\mathrm{Ker}(\varphi) \to \mathrm{Ran}(\varphi)$: $*$-isomorphism.

Hence, by (i), ψ : isometric $*$-isomorphism from $\mathcal{C}/\mathrm{Ker}(\varphi)$ to $\mathrm{Ran}(\varphi)$. But now, Since $\mathcal{C}/\mathrm{Ker}(\varphi)$ is a C^*-algebra, $\mathrm{Ran}(\varphi)$: C^*-subalgebra of $\mathcal{Q}$ ▨