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Exercise

Ex 11.6.

C, Q : C^*-algs. $\varphi : C \to Q$ $*$-hom.

(i). φ : isometric $\iff$ φ : injective.

(ii). $\ker\varphi \subset C$ is a C^*-sub alg and $\mathrm{Ran}\,\varphi \subset Q$ is a C^*-sub alg.

(i) lem. $\|\varphi(a)\|_Q \leq \|a\|_C$ for $\forall a \in C$.

$\because$) $\lambda \notin sp(a) \Rightarrow \varphi((\lambda 1_A - a)^{-1}(\lambda 1_A - a)) = \varphi(1_A) = 1_B$

$= \varphi((\lambda 1_A - a)^{-1})(\lambda 1_B - \varphi(a))$.

so $\lambda \notin sp(\varphi(a))$ and $sp(\varphi(a)) \subset sp(a)$.

since $r(\varphi(a)) \leq r(a)$, $\|\varphi(a)\|_Q \leq \|a\|_C$.

(ii). $\ker\varphi$ and $\mathrm{Ran}\,\varphi$ are ideals in C^*-algs. So we show only their completeness.

- $\ker\varphi$. φ is continuous by the lem above

  so $\{a_n\}_{n=1}^{\infty}$ convergent sequence in $\ker\varphi$.

  $\lim_{n\to\infty}\varphi(a_n) = \varphi(\lim_{n\to\infty} a_n) = 0$. so $\lim_{n\to\infty} a_n \in \ker\varphi$.

- $\mathrm{Ran}\,\varphi \simeq C/\ker\varphi$ and $C/\ker\varphi$ is a C^*-alg.

  so $\mathrm{Ran}\,\varphi$ is C^*-alg.

(i). ($\Rightarrow$). Since φ is isometric.

$\|\varphi(a)\|_Q = \|a\|_C$ f. ($\forall a \in C$)

$\varphi(a) = 0 \Rightarrow a = 0$. φ is injective.

($\Leftarrow$). φ^{-1} ; $\mathrm{Ran}\,\varphi \to C$ is also $*$-hom.

$\|a\|_C = \|\varphi^{-1}\circ\varphi(a)\|_C \leq \|\varphi(a)\|_Q \leq \|a\|_C$.

$\|a\|_C = \|\varphi(a)\|_Q$.

Ex 2.1.6 C: unital C^*-alg

(i) $\mathcal{U}_0(C)$ is a normal subgroup of $\mathcal{U}(C)$

(ii) $\mathcal{U}_0(C)$ is open and closed to $\mathcal{U}(C)$

(iii) $u \in C$ belongs to $\mathcal{U}_0(C)$

$\Leftrightarrow$ $u = e^{ia_1} e^{ia_2} \cdots e^{ia_n}$. for $\exists a_1, a_2 \cdots a_n \in C$ self adjoint elements.

∵ (i). $u_1, u_2 \in \mathcal{U}_0(C)$

$\Rightarrow \exists f_1 : [0,1] \to \mathcal{U}(C)$ $f_2 : [0,1] \to \mathcal{U}(C)$ conti.

s.t $f_i(0) = 1_C$ $f_i(1) = u_i$. $(i=1,2)$

then $f_1(t) f_2(t)$ is also conti.

so are $(f_1(t))^{-1}$.

So $\mathcal{U}_0(C)$ is a subgroup of $\mathcal{U}(C)$.

For $v \in \mathcal{U}(C)$. $g_1(t) = v f_1(t) v^*$

is homotopy between. 1_C and. $v u_1 v^*$.

So $v u v^* \in \mathcal{U}_0(C)$.

∴ $\mathcal{U}_0(C)$ is a normal subgroup of $\mathcal{U}(C)$.

(ii) Since $\mathcal{U}_0(C)$ is pathconnected,

$\mathcal{U}_0(C)$ is connected.

(iii). $G := \{ \exp(ia_1)\exp(ia_2) \cdots \exp(ia_n) \mid \exists n \in \mathbb{N}$, $\exists a_1, a_2, \cdots a_n \in C$ self adjoint $\}$.

$\forall u = \exp(ia_1) \cdots \exp(ia_n) \in G$.

$\forall v = \exp(ib_1) \cdots \exp(ib_m) \in G$

$uv \in G$.

$u^{-1} = \exp(-ia_n)\exp(-ia_{n-1}) \cdots \exp(-ia_1) \in G$.

So G is a group, and clearly. $G \subset \mathcal{U}_0(C)$.

For $u \in \mathcal{U}_0(C)$ and $v \in G$. s.t $\|u - v\| < 2$.

$\|1 - uv^*\| = \| v - u\| \|v^*\| < 2$.

(∵ $v \in G \subset \mathcal{U}_0(C)$, $1 = \|vv^*\| = \|v^{**}v^*\| = \|v^*\|^2$.).

By the lem 2.1.2 (iii). and its proof, $\exists a \in C$ self adjoint

$uv^* = \exp(ia)$ $u = \exp(ia) v \in G$.

So G is open in $\mathcal{U}(C)$.

and $\exists \{u_i\} \subset \mathcal{U}(C)$

$\mathcal{U}(C) = \bigsqcup G u_i$.

Since $G u_i \simeq G$ homeo.,

the complement of G $\bigsqcup_i G u_i \setminus G$ is open

So G is closed. since $G \neq \emptyset$,

G is one of components of $\mathcal{U}(C)$

∴ $G = \mathcal{U}_0(C)$

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Ex1.1.6 Correction

lem A, B: C^*-algebras.

$\varphi : A \to B$ $*$-homomorphism

Then $\|\varphi(a)\|_B \le \|a\|_A \quad \forall a \in A$.

$\lambda \notin sp(a) \Rightarrow \varphi((\lambda 1_A - a)^{-1}(\lambda 1_A - a)) = 1_B$

$\varphi((\lambda 1_A - a)^{-1})(\lambda 1_B - \varphi(a))$

$\lambda \notin sp(\varphi(a))$

Therefore $sp(\varphi(a)) \subset sp(a)$. $\forall a \in A$.

So $r(\varphi(a)) \le r(a) \quad \forall a \in A$.

In particular, if a is self adjoint element

$\varphi(a)^* = \varphi(a^*) = \varphi(a)$. $\varphi(a)$ is also self adjoint.

Therefore $r(\varphi(a)) = \|\varphi(a)\|_B \quad r(a) = \|a\|_A$

$\|\varphi(a)\|_B \le \|a\|_A$.

Now aa^* $(\forall a \in A)$ is self adjoint.

So. $\|\varphi(aa^*)\| = \|\varphi(a)\varphi(a)^*\| = \|\varphi(a)\|^2 \le \|aa^*\| = \|a\|^2$.

$\therefore \|\varphi(a)\|_B \le \|a\|_A$.

Ex2.1.6 Correction

(ii) If $u \in \mathcal{U}_0(C)$

by the lemma. 2.1.3 (iii) $(u, v \in \mathcal{U}(C),\ \|u-v\| < 2 \Rightarrow u \sim_h v)$

$B_u = \{w \in \mathcal{U}(C) \mid \|u-v\| < 2\} \subset \mathcal{U}_0(C)$.

So $\mathcal{U}_0(C)$ is open set.

If $u \notin \mathcal{U}_0(C)$ and $u \in \mathcal{U}(C)$,

$\Rightarrow B_u \cap \mathcal{U}_0(C) = \phi$

because if $B_u \cap \mathcal{U}_0(C) \ni w$,

$1_C \sim_h w \sim_h u$. contradiction.

So. $\mathcal{U}(C) \setminus \mathcal{U}_0(C)$ is open

$\mathcal{U}_0(C)$ is closed in $\mathcal{U}(C)$.

Ex 2.2.2. C: unital C^*-alg
$p \in \mathcal{P}(C)$

Then, $\sigma(p) \subset \{0,1\}$.

∵ $\lambda \in \sigma(p) \Rightarrow (\lambda 1_C - p)^{-1}$ does not exist.

So $\exists v \in C$. $v \neq 0$. $(\lambda 1_C - p)v = 0$

$\lambda v - pv = 0$. $\qquad pv = \lambda v$.

$p^2 v = p(pv) = \lambda pv = \lambda^2 v = \lambda v$.

$(\lambda^2 - \lambda)v = 0$. $\qquad \lambda(\lambda - 1)v = 0$.

$\therefore \lambda = 0$ or 1.

Ex2.2.3 $\mathcal{C}$: C^*-alg.

(i) $v \in C$. v^*v : projection

$\Rightarrow$ vv^* is also a projection.

(ii) Murray-von Neumon equivalent is transitive.

(i) Set $p = v^*v$. projection

Then $z = (1 - vv^*)v$ satisfies.

$z^*z = v^*(1 - vv^*)(1 - vv^*)v$ q^3

$= v^*(1 - 2vv^* + vv^*vv^*)v$

$= v^*v - 2p^2 + p^3 = p - 2p^2 + p^3 = 0$.

$\|z^*z\| = \|z\|^2 = 0$

$\therefore$ $z = 0$, i.e $v = vv^*v = vp = qv$.

Therefore $q^2 = vv^*vv^* = vpv^* = vv^* = q$.

$q^* = (vv^*)^* = vv^*$

q is a projection.

(ii) $p \sim q$, $q \sim r$

$\Rightarrow$ $p = v^*v$ $q = vv^*$ $\exists v \in C$

$q = w^*w$ $r = ww^*$ $\exists w \in C$.

set $z = wv$.

Then $z^*z = v^*w^*wv = v^*qv = v^*v = p$.

$zz^* = wvv^*w^* = wqw^* = ww^* = r$.

$\therefore$ $p \sim r$. //

Ex 3.14. $p, q \in \mathcal{P}(M_n(\mathbb{C}))$

(i) $p \sim q$

$\Leftrightarrow$ (ii) $tr(p) = tr(q)$

$\Leftrightarrow$ (iii) $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$

○ (i) $\Rightarrow$ (ii). $p \sim q$

$\Rightarrow \exists v \in M_n(\mathbb{C})$. $\quad p = v^*v \quad q = vv^*$

$tr(p) = tr(v^*v) = tr(vv^*) = tr(q)$.

(ii) $\Rightarrow$ (iii). Since $\sigma(p) \subset \{0,1\}$ and $p = p^*$,

$\exists\, g_1 \in GL_n(\mathbb{C}) \quad g_1 p g_1^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$.

$\exists\, g_2 \in GL_n(\mathbb{C}) \quad g_2 q g_2^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$

$tr(p) = tr(q)$.

$\Rightarrow$. $\dim p(\mathbb{C}^n) = \text{rank } p = tr(p) = tr(q) = \dim q(\mathbb{C}^n)$

(iii) $\Rightarrow$ (i) since p is a projection.

$\mathbb{C}^n = p(\mathbb{C}^n) \oplus \ker p$.

Therefore $\mathbb{C}^n = p(\mathbb{C}^n) \cap q(\mathbb{C}^n) \oplus p(\mathbb{C}^n) \cap \ker q$

$\oplus \ker p \cap q(\mathbb{C}^n) \oplus \ker p \cap \ker q$.

Take a basis of $\mathbb{C}^n$ $\{e_1, \dots, e_n\}$

where $\{e_1, \dots, e_{n_1}\}$ is a basis of $p(\mathbb{C}^n) \cap q(\mathbb{C}^n)$

$\{e_{n_1+1}, \dots, e_{n_2}\}$ 〃 $p(\mathbb{C}^n) \cap \ker q$

$\{e_{n_2+1}, \dots, e_{n_3}\}$ 〃 $\ker p \cap q(\mathbb{C}^n)$

$\{e_{n_3+1}, \dots, e_n\}$ 〃 $\ker p \cap \ker q$.

Since $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$.

and $p(\mathbb{C}^n) + \ker q = p(\mathbb{C}^n) \oplus \ker q \cap \ker p$,

$\dim(p(\mathbb{C}^n) \cap \ker q) = \dim(p(\mathbb{C}^n)) + \dim(\ker q) - \dim(p(\mathbb{C}^n) + \ker q)$

$= \dim(p(\mathbb{C}^n)) + \dim(\ker q) - \dim(p(\mathbb{C}^n) \oplus \ker q \cap \ker p)$

$= \dim(\ker q) - \dim(\ker q \cap \ker p)$

$= \dim(\ker p) - \dim(\ker q \cap \ker p)$

$= \cdots = \dim(q(\mathbb{C}^n) \cap \ker p)$.

$\therefore\ n_3 - n_2 = n_2 - n_1\ (=: d)$.

Therefore we can define $f : \mathbb{C}^n \to \mathbb{C}^n$ linear by

$$f(e_i) = \begin{cases} e_i & 1 \le i \le n_1 \\ e_{d+i} & n_1+1 \le i \le n_2 \\ 0 & n_2+1 \le i \le n_3. \\ 0 & n_3+1 \le i < n. \end{cases}$$

i.e.

$$f(e_1, \dots, e_n) = (e_1, \dots, e_n) \underbrace{\begin{pmatrix} 1 & & & & O \\ & \ddots & & & \\ O & & 0 & & \\ & 1 & & \ddots & \\ & & 0 & & 0 \end{pmatrix}}_{Q}.$$

Then

$f^*(e_1, \dots, e_n) = (e_1, \dots, e_n)\, Q^*$.

i.e.

$$f^*(e_i) = \begin{cases} e_i & 1 \le i \le n_1 \\ 0 & n_1+1 \le i \le n_2. \\ e_{i-d}. & n_2+1 \le i \le n_3 \\ 0 & n_3+1 \le i \le n. \end{cases}$$

$$f^* \circ f(e_1 \cdots e_n) = (e_1 \cdots e_n)\begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}$$

$$f \circ f^*(e_1 \cdots e_n) = (e_1 \cdots e_n)\begin{pmatrix} 1 & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ 0 & & & & & & 0 \end{pmatrix}$$

Set $\{s_1, \cdots s_n\}$ standard basis of $\mathbb{C}^n$

and $(s_1, \cdots s_n) = (e_1, \cdots e_n)R$ basis transformation.

Take $v = R^{-1}QR$

Then $p = v^*v$ and $q = vv^*$. //

- $D(\mathbb{C}) \simeq \mathbb{Z}_+$

$\because$ We define map

$tr : D(\mathbb{C}) \to \mathbb{Z}_+$

$tr([p]_D) = tr(p)$

Then For $p \in P_n(\mathbb{C})$, $q \in P_m(\mathbb{C})$ $(n \leq m)$

$\Rightarrow p \sim_0 q \Leftrightarrow p \oplus 0_{m-n} \sim q \Leftrightarrow tr(p \oplus 0_{m-n}) = tr(p) = tr(q)$

$(= \dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^m))$.

So tr is well defined.

Moreover $tr([p]_D + [q]_D) = tr([p \oplus q]_D)$

$= tr(p \oplus q) = tr(p) + tr(q) = tr([p]_D) + tr([q]_D)$

$\therefore$ tr is semigroup homomorphism.

tr is surjective because

$\forall n \in \mathbb{Z}_+$ $E_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in M_n(\mathbb{C})$.

$tr([E_n]_D) = n$.

tr is injective because.

if $tr([p]_D) = 0$

$\Rightarrow tr\ p = 0. \Leftrightarrow \dim p(\mathbb{C}^n) = 0$.

$\therefore [p]_D = [0]_D$. //

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Ex 3.1.5 $\mathcal{H}$: infinite Hilbert sp. $p, q \in P(B(\mathcal{H}))$

(i) $p \sim q \Leftrightarrow \dim(p(\mathcal{H})) = \dim(q(\mathcal{H}))$

(ii). $p \sim_u q \Leftrightarrow \dim(p(\mathcal{H})) = \dim(q(\mathcal{H}))$

(iii). $D(B(\mathcal{H})) \simeq \mathbb{Z}_+ \cup \{\infty\}$

with the adition $+$ in $\mathbb{Z}_+$ and $n+\infty = \infty + n = \infty$. for $\forall n \in \mathbb{Z}_+$

∵ (i)

$p \sim q \Rightarrow \exists v \in B(\mathcal{H})$.

$p = v^* v \quad q = v v^*$

Then for $\forall x \in \mathcal{H}$

$\forall e \in \ker v$. $\langle e, v^*(x) \rangle = \langle v(e), x \rangle = 0$.

∴ $(\ker v)^\perp \supset \operatorname{Ran} v^*$.

$x \in (\ker v)^\perp \Rightarrow$.

$\exists x_1 \in (\ker v)^\perp \cap (\operatorname{Ran} v^*) \quad \exists x_2 \in (\ker v)^\perp \cap (\operatorname{Ran} v^*)^\perp$

$x = x_1 + x_2$

$\forall w \in \mathcal{H}$ $\langle x_2, v^*(w) \rangle = \langle v(x_2), w \rangle = 0$

∴ $v(x_2) = 0 \quad x_2 \in \ker v \quad \cdots \quad x_2 = 0$

$x = x_1 \in (\ker v)^\perp \cap (\operatorname{Ran} v^*) \subset \operatorname{Ran} v^*$.

∴ $(\ker v)^\perp = \operatorname{Ran} v^*$.

We can say $(\ker v^*)^\perp = \operatorname{Ran} v$ in the same way.

$v|_{(\ker v)^\perp} : (\ker v)^\perp \to \operatorname{Ran} v$

$v^*|_{(\ker v^*)^\perp} : (\ker v^*)^\perp \to \operatorname{Ran} v^*$.

are isomorphisms.

Therefore $\mathrm{id}_{\operatorname{Ran} v^*} = v^* v|_{(\ker v)^\perp} : \underset{\operatorname{Ran} v^*}{\underset{\shortparallel}{(\ker v)^\perp}} \to \operatorname{Ran} v^*$.

$\mathrm{id}_{\operatorname{Ran} v} = v v^*|_{(\ker v^*)^\perp} : \underset{\operatorname{Ran} v}{\underset{\shortparallel}{(\ker v^*)^\perp}} \to \operatorname{Ran} v$.

∴ $\dim(p(\mathcal{H})) = \dim(q(\mathcal{H}))$

($\Leftarrow$) Since $\mathcal{H}$ is separable and we can take countable basis,

We can proof this in the same way of Exersize 3.1.4 of (iii) $\Rightarrow$ (i).

(ii). $p \sim_u q \Rightarrow \exists u \in \mathcal{U}(\mathbb{C})$

$q = u p u^*$ since u is invertible

$u(\mathcal{H}) = \mathcal{H}$. $u^*(\mathcal{H}) = \mathcal{H}$.

$\dim(q(\mathcal{H})) = \dim(u p u^*(\mathcal{H})) = \dim(p(\mathcal{H}))$

and $\dim(q(\mathcal{H})^\perp) = \dim((u p u^*(\mathcal{H}))^\perp) = \dim(p(\mathcal{H})^\perp)$

$\dim(p(\mathcal{H})) = \dim(q(\mathcal{H}))$ and $\dim(p(\mathcal{H})^\perp) = \dim(q(\mathcal{H})^\perp)$

$\Rightarrow$ We decompose $\mathcal{H}$ by

$p(\mathcal{H}) \cap q(\mathcal{H}) \oplus p(\mathcal{H}) \cap q(\mathcal{H})^\perp \oplus p(\mathcal{H})^\perp \cap q(\mathcal{H}) \oplus p(\mathcal{H})^\perp \cap q(\mathcal{H})^\perp$

$p(\mathcal{H}) = p(\mathcal{H}) \cap q(\mathcal{H}) \oplus p(\mathcal{H}) \cap q(\mathcal{H})^\perp$.

$q(\mathcal{H}) = p(\mathcal{H}) \cap q(\mathcal{H}) \oplus p(\mathcal{H})^\perp \cap q(\mathcal{H})^\perp$.

So $\dim(p(\mathcal{H}) \cap q(\mathcal{H})^\perp) = \dim(p(\mathcal{H})^\perp \cap q(\mathcal{H}))$,

We take a basis

$\{\phi_i^1\}$ of $p(\mathcal{H}) \cap q(\mathcal{H})$

$\{\phi_i^2\}$ of $p(\mathcal{H}) \cap q(\mathcal{H})^\perp$

$\{\phi_i^3\}$ of $p(\mathcal{H})^\perp \cap q(\mathcal{H})$

$\{\phi_i^4\}$ of $p(\mathcal{H})^\perp \cap q(\mathcal{H})^\perp$.

and take $u : \mathcal{H} \to \mathcal{H}$. unitary.

$$u(\phi_i^j) = \begin{cases} \phi_i^j & (j = 1, 4) \\ \phi_i^3 & (j = 2) \\ \phi_i^2 & (j = 3) \end{cases}$$

Then $q = u p u^*$. $q \sim_u p$

Ex 3.15

(iii) We cleam first

$M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$.

$\because$) It is clear $M_n(B(\mathcal{H})) \subset B(\mathcal{H})$

Let $\alpha_i : \mathcal{H}^n \to \mathcal{H}$ projection into i-th components.

$\beta_i : \mathcal{H} \to \mathcal{H}^n$ embedding into i-th components.

$\forall f \in B(\mathcal{H}^n)$

$f_{ij} := \alpha_i \circ f \circ \beta_j$.

then $(f_{ij})_{i,j=1,2,\cdots n} \in B(\mathcal{H}^n)$.

So. $M_n(B(\mathcal{H})) = B(\mathcal{H}^n)$.

We define a map.

$\dim : P(B(\mathcal{H})) \to \mathbb{Z}_+ \cup \{\infty\}$.

by $\dim(p) = \dim(p(\mathcal{H}^n))$

for $p \in P_n(B(\mathcal{H})) = P(B(\mathcal{H}^n))$

dim satisfies.

$\dim(p \oplus q) = \dim\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \dim\left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}(\mathcal{H}^{m+n})\right)$

$= \dim(p(\mathcal{H}^n)) + \dim(q(\mathcal{H}^m))$

for $p \in P_n(B(\mathcal{H}))$
$q \in P_m(B(\mathcal{H}))$

$p \in P_n(B(\mathcal{H}))$ $q \in P_m(B(\mathcal{H}))$ $n \leq m$.

$\Rightarrow$ $p \sim_0 q \iff p \oplus 0_{m-n} \sim q \iff \dim(p) = \dim(q)$.

by (i).

dim induce an isomorphism.

$\widetilde{\dim} : D(B(\mathcal{H})) \xrightarrow{\sim} \mathbb{Z}_+ \cup \{\infty\}$.

because.

dim is surjective.

and $\dim p = \dim q \iff p \sim_0 q$.

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Ex 3.3.3 $\mathcal{C}, \mathcal{Q}$: C^*-algs.

(i) $\varphi, \psi : \mathcal{C} \to \mathcal{Q}$ homotopic $*$-hom

$\Rightarrow K_0(\varphi) = K_0(\psi)$

(ii) $\mathcal{C}, \mathcal{Q}$: homotopy equivalent

$\Rightarrow K_0(\mathcal{C}) \cong K_0(\mathcal{Q})$

$\because$ (i) Let $\Phi : [0,1] \times \mathcal{C} \to \mathcal{Q}$ homotopy between φ and ψ.

i.e. Φ is continuous map

s.t. $\Phi(0,p) = \varphi(p)$ and $\Phi(1,p) = \psi(p)$.

Then $\forall p \in P_\infty(\mathcal{C})$.

$\Phi(t,p)$ is homotopy between $\varphi(p)$ and $\psi(p)$.

So $\varphi(p) \sim_h \psi(p) \Rightarrow [\varphi(p)]_0 = [\psi(p)]_0$.

$\therefore K_0(\varphi)([p]_0) = [\varphi(p)]_0 = [\psi(p)]_0 = K(\psi)([p]_0)$.

(ii). Since $\mathcal{C}$ and $\mathcal{Q}$ are homotopy equivalent,

$\exists \varphi : \mathcal{C} \to \mathcal{Q}$ $\exists \psi : \mathcal{Q} \to \mathcal{C}$.

s.t. $\psi \circ \varphi \sim_h id_{\mathcal{C}}$, $\varphi \circ \psi \sim_h id_{\mathcal{Q}}$.

By (i). $id_{K(\mathcal{C})} = K_0(id_{\mathcal{C}}) = K_0(\psi \circ \varphi)$

$= K_0(\psi) \circ K_0(\varphi)$.

and $id_{K_0(\mathcal{Q})} = K_0(\varphi) \circ K_0(\psi)$. //

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Ex 3.4.4

(i). $\forall n \in \mathbb{N}$.

$K_0(M_n(\mathbb{C})) \simeq \mathbb{Z}$.

(ii) $\mathcal{H}$: infinite dimensional. Hilbert sp.

Then

$K_0(\mathcal{B}(\mathcal{H})) \simeq \{0\}$.

(iii) Ω: compact connected Hausdorff sp.

Then $\exists$ $\dim: K_0(C(\Omega)) \to \mathbb{Z}$ surjective group hom

s.t. $\forall p \in \mathcal{P}_\infty(C(\Omega))$, $\forall x \in C(\Omega)$

$\dim([p]_0) = tr(p(x))$

$\because$ (i) Define a map $K_0(tr): K_0(M_n(\mathbb{C})) \to \mathbb{Z}$. by

for $p, q \in M_k(M_n(\mathbb{C})) = M_{kn}(\mathbb{C})$

$K_0(tr)([p]_0 - [q]_0) = tr(p) - tr(q)$

$= \dim(p(\mathbb{C}^{kn})) - \dim(q(\mathbb{C}^{kn}))$.

If $K_0(tr)([p]_0 - [q]_0) = 0$

$\Rightarrow \dim p(\mathbb{C}^{kn}) = \dim q(\mathbb{C}^{kn})$

$\Rightarrow p \sim q$. $\Rightarrow [p]_0 = [q]_0$.

$\therefore K_0(tr)$ is injective.

$K_0(tr)$ is surjective because

for $e = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} \in \mathcal{P}(M_n(\mathbb{C}))$,

$K_0(tr)([e]_0) = 1$. //

(ii). We claim first.

$M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^n)$

It is clear $M_n(\mathcal{B}(\mathcal{H})) \subset \mathcal{B}(\mathcal{H}^n)$.

and $\forall f \in \mathcal{B}(\mathcal{H}^n)$

define $f_{ij} \in \mathcal{B}(\mathcal{H})$

by $f_{ij}(v) := P_j \circ f \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \} i$.

where $P_j: \mathcal{H}^n \to \mathcal{H}$ is projection into j th component.

$(f_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$.

Therefore $M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^n)$

We define a map $\dim: \mathcal{P}_\infty(\mathcal{B}(\mathcal{H})) \to \mathbb{Z}_+ \cup \{\infty\}$.

by

$\dim(p) = (p(\mathcal{H}^n))$ for $p \in \mathcal{P}_n(\mathcal{B}(\mathcal{H})) = \mathcal{P}(\mathcal{B}(\mathcal{H}^n))$.

This map is surjective

and for $p \in \mathcal{P}_n(\mathcal{B}(\mathcal{H}))$ and $q \in \mathcal{P}_m(\mathcal{B}(\mathcal{H}))$ $(n \le m)$

$\dim(p \oplus q) = \dim\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \dim\left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}(\mathcal{H}^{m+n})\right)$

$= \dim(p(\mathcal{H}^m)) + \dim q(\mathcal{H}^m)$

and $p \sim q \iff p \oplus 0_{m-n} \sim q \iff \dim(p) = \dim(q)$.

So $\dim$ Induces the map

$d: \mathcal{D}(\mathcal{B}(\mathcal{H})) \to \mathbb{Z}_+ \cup \{\infty\}$

$d([p]_0) = \dim(p)$,

$\therefore G(\mathcal{D}(\mathcal{B}(\mathcal{H}))) = \{0\}$

Ex 3.4.4 (ii). Correction.

(I). $M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^n)$

$\because$

It is clear that $M_n(\mathcal{B}(\mathcal{H})) \subset \mathcal{B}(\mathcal{H}^n)$

Let $p_i : \mathcal{H}^n \to \mathcal{H}$.

projection into i-th component.

$q_j : \mathcal{H} \to \mathcal{H}^n$ embedding into j-th component.

For $\forall f \in \mathcal{B}(\mathcal{H}^n)$, we define

$$f_{ij}(v) = p_j \circ f \circ q_i(v) \quad (v \in \mathcal{H})$$

Then $(f_{ij})_{i,j=1,2\cdots n} \in M_n(\mathcal{B}(\mathcal{H}))$

$\therefore$ $M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^n)$

(II). $\mathbb{D}(\mathcal{B}(\mathcal{H})) \cong \mathbb{Z}_+ \cup \{\infty\}$

$\because$ Exercise 3.1.5

(III) $\mathcal{G}(\mathbb{Z}_+ \cup \{\infty\}) = \{0\}$

$\because$ $\forall \langle x, y \rangle, \langle z, w \rangle \in \mathcal{G}(\mathbb{Z}_+ \cup \{\infty\})$

$x + w + \infty = z + y + \infty = \infty$

$\langle x, y \rangle = \langle z, w \rangle$.

So $\mathcal{G}(\mathbb{Z}_+ \cup \{\infty\}) = \{0\}$