HW7

[2.4 : 2, 6, 12, 20, 22, 23]

2. $\vec{f}(x,y) = (e^{x+y}, xe^y)$, $\vec{g}(x,y) = (\ln(xy), ye^x)$.

$$D\vec{f}(x,y) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^y & xe^y \end{pmatrix} \qquad D\vec{g}(x,y) = \begin{pmatrix} \frac{1}{x} & \frac{1}{y} \\ ye^x & e^x \end{pmatrix}$$

$$D\vec{f}(x,y) + D\vec{g}(x,y) = \begin{pmatrix} e^{x+y} + \frac{1}{x} & e^{x+y} + \frac{1}{y} \\ e^y + ye^x & xe^y + e^x \end{pmatrix}$$

$$D(\vec{f}+\vec{g})(x,y) = D(e^{x+y} + \ln(xy),\ xe^y + ye^x)$$
$$= \begin{pmatrix} e^{x+y} + \frac{1}{x} & e^{x+y} + \frac{1}{y} \\ e^y + ye^x & xe^y + e^x \end{pmatrix} = D\vec{f}(x,y) + D\vec{g}(x,y)$$

6. $f(x,y) = e^{xy}$, $g(x,y) = x\sin 2y$.

$Df(x,y) = (ye^{xy}, xe^{xy})$; $Dg(x,y) = (\sin 2y, 2x\cos 2y)$

$g(x,y)Df(x,y) = (xye^{xy}\sin 2y, x^2e^{xy}\sin 2y)$; $f(x,y)Dg(x,y) = (e^{xy}\sin 2y, 2xe^{xy}\cos 2y)$

$$Dfg(x,y) = D(xe^{xy}\sin 2y)$$
$$= (e^{xy}\sin 2y + xye^{xy}\sin 2y,\ x^2e^{xy}\sin 2y + 2xe^{xy}\cos 2y)$$
$$= (e^{xy}\sin 2y, 2xe^{xy}\cos 2y) + (xye^{xy}\sin 2y, x^2e^{xy}\sin 2y)$$
$$= f(x,y)Dg(x,y) + g(x,y)Df(x,y)$$

Note that $g(x,y) = x\sin 2y \neq 0$ for $(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \text{ or } y \neq n\frac{\pi}{2}, \forall n \in \mathbb{Z}\}$.

Thus, for $x \neq 0$, $y \neq n\frac{\pi}{2}$ for all $n \in \mathbb{Z}$,

$$D\frac{f}{g}(x,y) = D\left(\frac{e^{xy}}{x\sin 2y}\right) = \left(\frac{(x\sin 2y)(ye^{xy}) - e^{xy}\sin 2y}{x^2\sin^2 2y}, \frac{(x\sin 2y)(xe^{xy}) - 2xe^{xy}\cos 2y}{x^2\sin^2 2y}\right)$$
$$= \left(\frac{xye^{xy}\sin 2y - e^{xy}\sin 2y}{x^2\sin^2 2y}, \frac{x^2e^{xy}\sin 2y - 2xe^{xy}\cos 2y}{x^2\sin^2 2y}\right)$$
$$= \frac{1}{x^2\sin^2 2y}\left((xye^{xy}\sin 2y, x^2e^{xy}\sin 2y) - (e^{xy}\sin 2y, 2xe^{xy}\cos 2y)\right)$$
$$= \frac{1}{g^2(x,y)}\left(g(x,y)Df(x,y) - f(x,y)Dg(x,y)\right)$$
$$= \frac{g(x,y)Df(x,y) - f(x,y)Dg(x,y)}{g^2(x,y)}$$

12. $f(x,y) = \sin\sqrt{x^2+y^2} = \sin\left((x^2+y^2)^{1/2}\right)$

$$f_x(x,y) = \cos\left((x^2+y^2)^{1/2}\right) \cdot \tfrac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x = \frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{x\cos\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}$$

$$f_y(x,y) = \cos\left((x^2+y^2)^{1/2}\right) \tfrac{1}{\cancel{2}}(x^2+y^2)^{-1/2}\, \cancel{2}y = \frac{y\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{y\cos\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}$$

$$f_{xx}(x,y) = \frac{\left(\cos\left((x^2+y^2)^{1/2}\right) - \dfrac{x^2\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}\right)(x^2+y^2)^{1/2} - \dfrac{x^2\cos\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}}{x^2+y^2}$$

$$= \frac{y^2\cos\left((x^2+y^2)^{1/2}\right) - x^2(x^2+y^2)^{1/2}\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{3/2}}$$

$$f_{yy}(x,y) = \frac{x^2\cos\left((x^2+y^2)^{1/2}\right) - y^2(x^2+y^2)^{1/2}\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{3/2}}$$

//

$$f_{xy}(x,y) = \frac{-xy\dfrac{\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}(x^2+y^2)^{1/2} - \dfrac{x\cos\left((x^2+y^2)^{1/2}\right)\cdot y}{(x^2+y^2)^{1/2}}}{x^2+y^2}$$

$$= -xy\frac{\dfrac{\cos\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}} + \sin\left((x^2+y^2)^{1/2}\right)}{x^2+y^2} = -xy\frac{\cos\left((x^2+y^2)^{1/2}\right) + (x^2+y^2)^{1/2}\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{3/2}}$$

$$f_{yx}(x,y) = -xy\frac{\cos\left((x^2+y^2)^{1/2}\right) + (x^2+y^2)^{1/2}\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{3/2}}$$

//

20. (a) $f(x,y,z) = x^2+y^2-2z^2$. $f_x(x,y,z) = 2x$; $f_y(x,y,z) = 2y$; $f_z(x,y,z) = -4z$

$f_{xx}(x,y,z) = 2$; $f_{yy}(x,y,z) = 2$; $f_{zz}(x,y,z) = -4$;

$f_{xy}(x,y,z) = f_{yx}(x,y,z) = f_{xz}(x,y,z) = f_{zx}(x,y,z) = f_{yz}(x,y,z) = f_{zy}(x,y,z) = 0$.

Thus, f is C^2 for all its second-order derivatives are constant functions, which are continuous and since $f_{xx}(x,y,z) + f_{yy}(x,y,z) + f_{zz}(x,y,z) = 2+2-4=0$, f is harmonic.

- $f(x,y,z) = x^2 - y^2 + z^2$ $\quad f_x(x,y,z) = 2x$; $f_y(x,y,z) = -2y$; $f_z(x,y,z) = 2z$.

$f_{xx}(x,y,z) = 2$; $\quad f_{yy}(x,y,z) = -2$; $\quad f_{zz}(x,y,z) = 2$;

$f_{xy}(x,y,z) = f_{yx}(x,y,z) = f_{xz}(x,y,z) = f_{zx}(x,y,z) = f_{yz}(x,y,z) = f_{zy}(x,y,z) = 0$.

Since all of the second order derivatives are constant functions, f is C^2. However, $f_{xx}(x,y,z) + f_{yy}(x,y,z) + f_{zz}(x,y,z) = 2-2+2 = 2 \neq 0$, thus f is not harmonic.

(b) Consider $g(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_{n-1}^2 - (n-1)x_n^2$.

$g_{x_1} = 2x_1$; $g_{x_2} = 2x_2$; $\ldots$; $g_{x_{n-1}} = 2x_{n-1}$, $g_{x_n} = -2(n-1)x_n$.

$g_{x_1x_1} = 2$, $g_{x_2x_2} = 2$, $\ldots$, $g_{x_{n-1}x_{n-1}} = 2$, $g_{x_nx_n} = -2(n-1)$.

Since g_{x_i} is a function of x_i and does not depend on all the other x_j's $(j \neq i)$,

$g_{x_ix_j} = 0$ when $i \neq j$ $\quad (i,j = 1,2,\ldots,n)$.

Since all the second-order derivatives are constant functions, they are all continuous, thus g is C^2. Furthermore,

$$g_{x_1x_1} + g_{x_2x_2} + \ldots + g_{x_{n-1}x_{n-1}} + g_{x_nx_n} = \underbrace{2+2+\ldots+2}_{n-1} - 2(n-1)$$

$$= 2(n-1) - 2(n-1) = 0.$$

Hence g is harmonic.

22. $f(x,y) = \begin{cases} xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right) & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

(a) Let $(x,y) \neq (0,0)$.

$$f_x(x,y) = y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{2x(\cancel{x^2}+y^2) - 2x(\cancel{x^2}-y^2)}{(x^2+y^2)^2}\right) = \frac{y(x^2-y^2)}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2}$$

$$f_y(x,y) = x\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{-2y(x^2+\cancel{y^2}) - 2y(x^2-\cancel{y^2})}{(x^2+y^2)^2}\right) = \frac{x(x^2-y^2)}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2}$$

(b) $f_x(0,y) \overset{(a)}{=} \dfrac{y(-y^2)}{y^2} + \dfrac{0}{y^4} = -y$ (note that $y \neq 0$ since (a) holds for $(x,y) \neq (0,0)$) //

$f_y(x,0) \overset{(a)}{=} \dfrac{x(x^2)}{x^2} - \dfrac{0}{x^4} = x$ (note that $x \neq 0$ since (a) holds for $(x,y) \neq (0,0)$) //

(c) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$

$f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$

$f_{xy}(0,0) = \lim_{h\to 0} \dfrac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \dfrac{-h}{h} = \lim_{h\to 0} -1 = -1.$ //

$f_{yx}(0,0) = \lim_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \dfrac{h}{h} = \lim_{h\to 0} 1 = 1 \neq f_{xy}(0,0)$ //

We can check that f_x, f_y are continuous at $(0,0)$ but f_{xx}, f_{xy}, f_{yx}, f_{yy} are not continuous there. This agrees with the statement of Theorem 4.3.

23. Any given plane can be written as $Ax + By + Cz = D$ ... $(*)$ where $C \neq 0$.

We want to show that a plane is always a minimal surface, that is, it satisfies equation (14): $(1+z_y^2)z_{xx} + (1+z_x^2)z_{yy} = 2z_x z_y z_{xy}$.

From $(*)$, a plane can be rewritten as $z = \dfrac{D - Ax - By}{C}$

$z_x = -A/C$, $z_{xx} = 0$, $z_{xy} = 0$

$z_y = -B/C$, $z_{yy} = 0$, $z_{yx} = 0$.

$2z_x z_y z_{xy} = 0$

$(1+z_y^2)z_{xx} + (1+z_x^2)z_{yy} = (1+B^2/C^2)\cdot 0 + (1+A^2/C^2)\, 0 = 0 = 2z_x z_y z_{xy}$,

which verifies equation (14). Thus a plane is a minimal surface. //

[2.5 : 2, 4, 6, 10, 12]

2. $f(x,y) = \sin(xy)$, $x = s+t$, $y = s^2+t^2$.

(a) $f(s,t) = \sin\left((s+t)(s^2+t^2)\right) = \sin(s^3+s^2t+st^2+t^3)$

$$\frac{\partial f}{\partial s}(s,t) = (3s^2+2st+t^2)\cos(s^3+s^2t+st^2+t^3) \,//$$

$$\frac{\partial f}{\partial t}(s,t) = (s^2+2st+3t^2)\cos(s^3+s^2t+st^2+t^3) \,//$$

(b) $$\frac{\partial f}{\partial s}(s,t) = \frac{\partial f}{\partial x}(x,y)\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x,y)\frac{\partial y}{\partial s}(s,t)$$

$$= y\cos(xy)\cdot 1 + x\cos(xy)\; 2s = \left(s^2+t^2+2s(s+t)\right)\cos\left((s+t)(s^2+t^2)\right)$$

$$= (3s^2+2st+t^2)\cos(s^3+s^2t+st^2+t^3) \,//$$

$$\frac{\partial f}{\partial t}(s,t) = \frac{\partial f}{\partial x}(x,y)\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}(x,y)\frac{\partial y}{\partial t}(s,t)$$

$$= y\cos(xy)\cdot 1 + x\cos(xy)\cdot 2t = \left(s^2+t^2+2t(s+t)\right)\cos\left((s+t)(s^2+t^2)\right)$$

$$= (s^2+2st+3t^2)\cos(s^3+s^2t+st^2+t^3) \,//$$

4. $z = x^2+y^3$, $x = st$, $y = y(s,t)$.

$$\frac{\partial y}{\partial t}(2,1) = 0.$$

$$\frac{\partial z}{\partial t}(2,1) = \frac{\partial z}{\partial x}(x,y)\frac{\partial x}{\partial t}(s,t) + \frac{\partial z}{\partial y}(x,y)\frac{\partial y}{\partial t}(s,t)\Big|_{(s,t)=(2,1)}$$

$$= 2x\cdot s + 3y^2\frac{\partial y}{\partial t}(s,t)\Big|_{(s,t)=(2,1)} = 2s^2t + 3y^2\cdot\frac{\partial y}{\partial t}(s,t)\Big|_{(s,t)=(2,1)}$$

$$= 8 + 0 = 8 \,//$$

6. $V(x,y) = xy^2$, $x = x(t)$, $y = y(t)$.

$$\frac{dx}{dt}\Big|_{(x,y)=(6,1.5)} = -0.25 \text{ in/min}, \quad \frac{dy}{dt}\Big|_{(x,y)=(6,1.5)} = -0.125 \text{ in/min}.$$

$$\frac{dV}{dt}\Big|_{(x,y)=(6,1.5)} = \frac{\partial V}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}\Big|_{(x,y)=(6,1.5)} = y^2\frac{dx}{dt} + 2xy\frac{dy}{dt}\Big|_{(x,y)=(6,1.5)}$$

$$= (1.5)^2(-0.25) + 2(6)(1.5)(-0.125) = -0.5625 - 2.25 = -2.8125 \text{ in}^3/\text{min}. \,//$$

The butter is melting at 2.8125 in^3/min, i.e. the solid volume of butter is decreasing at 2.8125 in^3 per minute.

10. $z = f(x+y, x-y)$, $u = x+y$, $v = x-y$

$= f(u, v)$

Suppose that $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are continuous.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2.$$

12. $w = f\left(\frac{x^2-y^2}{x^2+y^2}\right)$, $u = \frac{x^2-y^2}{x^2+y^2}$, then $w = f(u)$.

Suppose that w is differentiable with respect to u, that is $\frac{dw}{du}$ exists.

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = x\left(\frac{dw}{du}\frac{\partial u}{\partial x}\right) + y\left(\frac{dw}{du}\frac{\partial u}{\partial y}\right)$$

$$= x\frac{dw}{du}\frac{2x(x^2+y^2) - (x^2-y^2)2x}{(x^2+y^2)^2} + y\frac{dw}{du}\frac{-2y(x^2+y^2) - (x^2-y^2)2y}{(x^2+y^2)^2}$$

$$= \frac{dw}{du}\cdot\frac{1}{(x^2+y^2)^2}\left(4x^2y^2 - 4x^2y^2\right) = 0\cdot\frac{dw}{du} = 0.$$