HW6

[2.3: 14, 16, 18, 22, 24, 34, 36, 50]

14. $f(x,y) = x^2y + e^{y/x}$, $\vec{a} = (1,0)$

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}))$$
$$= \left(2xy + e^{y/x}\left(-\frac{y}{x^2}\right)\Big|_{(x,y)=(1,0)},\ x^2 + e^{y/x}\frac{1}{x}\Big|_{(x,y)=(1,0)} \right) = (0, 1+1) = (0,2). //$$

16. $f(x,y,z) = \sin(xyz)$, $\vec{a} = (\pi, 0, \frac{\pi}{2})$

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$
$$= \left(yz\cos(xyz)\Big|_{(x,y,z)=(\pi,0,\frac{\pi}{2})},\ xz\cos(xyz)\Big|_{(x,y,z)=(\pi,0,\frac{\pi}{2})},\ xy\cos(xyz)\Big|_{(x,y,z)=(\pi,0,\frac{\pi}{2})} \right)$$
$$= \left(0, \frac{\pi^2}{2}, 0\right) //$$

18. $f(x,y) = e^{xy} + \ln(x-y)$, $\vec{a} = (2,1)$

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}))$$
$$= \left(ye^{xy} + \frac{1}{x-y}\Big|_{(x,y)=(2,1)},\ xe^{xy} - \frac{1}{x-y}\Big|_{(x,y)=(2,1)} \right) = (e^2+1, 2e^2-1). //$$

22. $\vec{f}(t) = (t, \cos 2t, \sin 5t)$, $a = 0$.

$$D\vec{f}(a) = \begin{pmatrix} \frac{dt}{dt}\Big|_{t=0} \\ \frac{d\cos 2t}{dt}\Big|_{t=0} \\ \frac{d\sin 5t}{dt}\Big|_{t=0} \end{pmatrix} = \begin{pmatrix} 1 \\ -2\sin 2t\Big|_{t=0} \\ 5\cos 5t\Big|_{t=0} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} //$$

24. $\vec{f}(x,y) = (x^2y, x+y^2, \cos(\pi xy))$, $\vec{a} = (2,-1)$

$$D\vec{f}(\vec{a}) = \begin{pmatrix} \frac{\partial}{\partial x}x^2y\Big|_{(x,y)=(2,-1)} & \frac{\partial}{\partial y}x^2y\Big|_{(x,y)=(2,-1)} \\ \frac{\partial}{\partial x}(x+y^2)\Big|_{(x,y)=(2,-1)} & \frac{\partial}{\partial y}(x+y^2)\Big|_{(x,y)=(2,-1)} \\ \frac{\partial}{\partial x}\cos(\pi xy)\Big|_{(x,y)=(2,-1)} & \frac{\partial}{\partial y}\cos(\pi xy)\Big|_{(x,y)=(2,-1)} \end{pmatrix} = \begin{pmatrix} 2xy\Big|_{(x,y)=(2,-1)} & x^2\Big|_{x=2} \\ 1 & 2y\Big|_{y=-1} \\ -\pi y\sin(\pi xy)\Big|_{(x,y)=(2,-1)} & -\pi x\sin(\pi xy)\Big|_{(x,y)=(2,-1)} \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{pmatrix} //$$

34. $f(2,3)=12$, $f(1.98,3)=12.1$, $f(2,3.01)=12.2$.

Then, $f_x(2,3) \approx \dfrac{f(1.98,3)-f(2,3)}{1.98-2} = \dfrac{12.1-12}{-0.02} = \dfrac{0.1}{-0.02} = -5$

and $f_y(2,3) \approx \dfrac{f(2,3.01)-f(2,3)}{3.01-3} = \dfrac{12.2-12}{0.01} = \dfrac{0.2}{0.01} = 20$.

(a) An approximate equation for the plane tangent to the graph at $(2,3,12)$ can be given by

$$z = f(2,3) + f_x(2,3)(x-2) + f_y(a,b)(y-3)$$

$$\approx 12 - 5(x-2) + 20(y-3) = 12 - 5x + 20y + 10 - 60 = -38 - 5x + 20y.$$

(b) We want to estimate $f(1.98, 2.98)$

Since $z = -38 - 5x + 20y$ is an approximate equation for the plane tangent to $f(x,y)$,

$f(1.98, 2.98)$ is approximately equal to $z\Big|_{(x,y)=(1.98,2.98)} = -38 - 9.9 + 59.6$

$= 11.7$ //

36. $f(x,y) = 3 + \cos(\pi x y)$. We want to approximate $f(0.98, 0.51)$.

(a) $Df(x,y) = \left(-\pi y \sin(\pi x y) \quad -\pi x \sin(\pi x y)\right)$

$$h(0.98, 0.51) = f(1, 0.5) + Df(1, 0.5)\begin{pmatrix} 0.98-1 \\ 0.51-0.5 \end{pmatrix}$$

$$= 3 + \cos\frac{\pi}{2} + \left(-\frac{\pi}{2}\sin\frac{\pi}{2} \quad -\pi \sin\frac{\pi}{2}\right)\begin{pmatrix} -0.02 \\ 0.01 \end{pmatrix}$$

$$= 3 + \left(-\frac{\pi}{2} \quad -\pi\right)\begin{pmatrix} -0.02 \\ 0.01 \end{pmatrix} = 3 + 0.01\pi - 0.01\pi = 3$$ //

(b) $f(0.98, 0.51) = 3 + \cos(0.4998\pi) \approx 3.00062832$.

$|f(0.98,0.51) - h(0.98,0.51)| < 0.0007 = 7 \times 10^{-4}$.

50. $g(x,y) = (xy)^{1/3}$.

(a) Since g is a composition of a product of two polynomials and an exponential function, g is continuous everywhere on its domain, and thus at $(0,0)$.

Specifically,

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} (xy)^{1/3} = \left(\lim_{(x,y)\to(0,0)} xy \right)^{1/3}$$

$$= \left(\lim_{x\to 0} x \cdot \lim_{y\to 0} y \right)^{1/3} = (0 \cdot 0)^{1/3} = 0 = g(0,0)$$ //

(b) Let $xy \neq 0$, then $x \neq 0$ and $y \neq 0$.

$$\frac{\partial g}{\partial x} = \frac{1}{3}(xy)^{-2/3}\, y = \frac{y}{3(xy)^{2/3}}$$ //

$$\frac{\partial g}{\partial y} = \frac{1}{3}(xy)^{-2/3}\, x = \frac{x}{3(xy)^{2/3}}$$ //

(c) $$g_x(0,0) = \lim_{h\to 0} \frac{g(0+h,0) - g(0,0)}{h} = \lim_{h\to 0} \frac{(h \cdot 0)^{1/3} - 0}{h} = \lim_{h\to 0} 0 = 0$$ //

$$g_y(0,0) = \lim_{h\to 0} \frac{g(0,0+h) - g(0,0)}{h} = \lim_{h\to 0} \frac{(0 \cdot h)^{1/3} - 0}{h} = \lim_{h\to 0} 0 = 0$$ //

(d) Since $\lim_{x\to 0} g_x(x,x^2) = \lim_{x\to 0} \frac{1}{3}x^2 \cdot x^{-2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}$ and

$\lim_{x\to 0} g_x(x,x) = \lim_{x\to 0} \frac{1}{3}x \cdot x^{-4/3} = \lim_{x\to 0} \frac{1}{3}x^{-1/3}$ does not exist,

$\lim_{(x,y)\to(0,0)} g_x(x,y)$ does not exist and $\frac{\partial}{\partial x} g(x,y)$ is not continuous at $(0,0)$. //

Similarly, $\lim_{y\to 0} g_y(y^2,y) = \lim_{y\to 0} \frac{1}{3}y^2 \cdot y^{-2} = \lim_{y\to 0} \frac{1}{3} = \frac{1}{3}$ and

$\lim_{y\to 0} g_y(y,y) = \lim_{y\to 0} \frac{1}{3}y \cdot y^{-4/3} = \lim_{y\to 0} \frac{1}{3}y^{-1/3}$ does not exist,

$\lim_{(x,y)\to(0,0)} g_y(x,y)$ does not exist and $\frac{\partial}{\partial y} g(x,y)$ is not continuous at $(0,0)$. //

Solutions to (e) and (f) are on the next page ⟶

(e), (f) From (c), $g_x(0,0) = g_y(0,0) = 0$. Furthermore, $g(0,0) = 0$.

Hence, $h(x,y) = g(0,0) + g_x(0,0)(x-0) + g_y(0,0)(y-0) = 0$.

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y)-h(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{(xy)^{1/3}}{\sqrt{x^2+y^2}}$$

but this limit does not exist.

$\because$ $\displaystyle\lim_{\substack{x\to 0\\ y=0}} \frac{(xy)^{1/3}}{\sqrt{x^2+y^2}} = \lim_{x\to 0} 0 = 0$

$\displaystyle\lim_{\substack{x\to 0^+\\ y=x}} \frac{(xy)^{1/3}}{\sqrt{x^2+y^2}} = \lim_{x\to 0^+} \frac{x^{2/3}}{\sqrt{x^2+x^2}} = \lim_{x\to 0^+} \frac{x^{2/3}}{\sqrt{2}\,|x|} = \lim_{x\to 0^+} \frac{1}{\sqrt{2}} x^{-1/3}$ does not exist.

(Especially, $\displaystyle\lim_{\substack{x\to 0\\ y=x}} \frac{(xy)^{1/3}}{\sqrt{x^2+y^2}}$ does not exist.) $\square$

Thus g is not differentiable at $(0,0)$ and so it has no tangent plane there.