HW 5

[2.2: 40, 46, 49]

40. $g(x,y) = \begin{cases} \dfrac{x^3+x^2+xy^2+y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$

When $(x,y) \neq (0,0)$, $x^2+y^2 \neq 0$ and both $x^3+x^2+xy^2+y^2$ and x^2+y^2 are continuous on $\mathbb{R}^2$.

Thus, the only possible discontinuity is at the origin $(0,0)$.

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2(x+1)+y^2(x+1)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x+1)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} (x+1) = 1 \neq 2 = g(0,0)$$

Hence, $g(x,y)$ is not continuous at the origin. //

46. $f(x,y) = 2x - 10y + 3$

(a) Let $\|(x,y) - (5,1)\| < \delta$, that is $0 < \sqrt{(x-5)^2+(y-1)^2} < \delta$.

Thus $(x-5)^2 + (y-1)^2 < \delta^2$ which implies $|x-5| < \delta$ and $|y-1| < \delta$.

(b) Let $\|(x,y) - (5,1)\| < \delta$, then

$$|f(x,y) - 3| = |2x-10y+3-3| = |2x-10y| = 2|x-5y| = 2|x-5+5-5y|$$

$$= 2|x-5+5(1-y)| \leq 2|x-5| + 10|y-1| \underset{(a)}{<} 2\delta + 10\delta = 12\delta \;//$$

(c) For any $\varepsilon > 0$, we take $\delta = \frac{1}{12}\varepsilon$, then from (c),

whenever $\|(x,y) - (5,1)\| < \delta$, we have $|f(x,y) - 3| < 12\delta = 12 \cdot \frac{1}{12}\varepsilon = \varepsilon$.

This implies $\lim_{(x,y)\to(5,1)} f(x,y) = 3$ //

49. (a) Let $a, b \in \mathbb{R}$.

Then, $(a-b)^2 \geq 0$ which implies $a^2 - 2ab + b^2 \geq 0$

$\Leftrightarrow a^2 + b^2 \geq 2ab$

Furthermore, $(a+b)^2 \geq 0$ which implies $a^2 + 2ab + b^2 \geq 0$

$\Leftrightarrow a^2+b^2 \geq -2ab$

Thus, $2ab, -2ab \leq a^2+b^2$ for any $a,b \in \mathbb{R}$. Hence, $2|ab| \leq a^2+b^2$ holds for all $a,b \in \mathbb{R}$.

(b) $f(x,y) = xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)$

Let $0 < \|(x,y)\| < \delta$, then $x^2+y^2 < \delta^2$.

Applying (a), we have

$$|f(x,y)| = \left|xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\right| \leq |xy|\,\frac{x^2+y^2}{x^2+y^2} = |xy| \overset{(a)}{\leq} \tfrac{1}{2}(x^2+y^2) < \tfrac{1}{2}\delta^2.$$

(c) Let $\varepsilon > 0$, put $\delta = \sqrt{2\varepsilon}$.

From (b), we have $|f(x,y)| < \frac{1}{2}\delta^2 = \varepsilon$ whenever $\|(x,y)\| < \delta$.

That is $\|(x,y) - (0,0)\| < \delta \Rightarrow |f(x,y) - 0| < \varepsilon$.

Thus, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

[2.3 : 4, 6, 12, 28, 30, 32.]

4. $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3-y^2}{1+x^2+3y^4}\right) = \frac{3x^2(1+x^2+3y^4) - (x^3-y^2)(2x)}{(1+x^2+3y^4)^2} = \frac{x^4+3x^2+9x^2y^4+2xy^2}{(1+x^2+3y^4)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3-y^2}{1+x^2+3y^4}\right) = \frac{-2y(1+x^2+3y^4) - (x^3-y^2)(12y^3)}{(1+x^2+3y^4)^2} = \frac{6y^5-12x^3y^3-2x^2y-2y}{(1+x^2+3y^4)^2}$$

6. $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\ln(x^2+y^2)) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\ln(x^2+y^2)) = \frac{2y}{x^2+y^2}$$

12. $$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(\sin(x^2y^3z^4)) = \cos(x^2y^3z^4)\cdot 2xy^3z^4$$
$$= 2xy^3z^4\cdot\cos(x^2y^3z^4)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(\sin(x^2y^3z^4)) = 3x^2y^2z^4\cdot\cos(x^2y^3z^4)$$

$$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(\sin(x^2y^3z^4)) = 4x^2y^3z^3\cdot\cos(x^2y^3z^4)$$

28. $\vec{f}(x,y) = \left(\dfrac{xy^2}{x^2+y^4}, \dfrac{x}{y} + \dfrac{y}{x} \right)$, domain $= \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x=0 \text{ or } y=0\}$.

(this will not be graded)

$$D\vec{f}(x,y) = \begin{pmatrix} \dfrac{y^2(x^2+y^4) - xy^2(2x)}{(x^2+y^4)^2} & \dfrac{2xy(x^2+y^4) - xy^2(4y^3)}{(x^2+y^4)^2} \\ \dfrac{1}{y} - \dfrac{y}{x^2} & -\dfrac{x}{y^2} + \dfrac{1}{x} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{y^6 - x^2y^2}{(x^2+y^4)^2} & \dfrac{2x^2y - 2xy^5}{(x^2+y^4)^2} \\ \dfrac{1}{y} - \dfrac{y}{x^2} & -\dfrac{x}{y^2} + \dfrac{1}{x} \end{pmatrix}$$

Each of the entries in the matrix is continuous on $\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 \mid x=0 \text{ or } y=0\}$, thus $\vec{f}(x,y)$ is differentiable at every point in its domain.

30. $z = 4\cos xy$, $z(\frac{\pi}{3}, 1) = 4\cos\frac{\pi}{3} = 2$

$z_x = -4y \sin xy$, $z_x(\frac{\pi}{3}, 1) = -4\sin\frac{\pi}{3} = -4(\frac{1}{2}\sqrt{3}) = -2\sqrt{3}$

$z_y = -4x\sin xy$, $z_y(\frac{\pi}{3}, 1) = -4(\frac{\pi}{3})\sin\frac{\pi}{3} = -\frac{4\pi}{3}\cdot\frac{1}{2}\sqrt{3} = -\frac{2}{3}\sqrt{3}\pi$.

Thus

$$z = z(\tfrac{\pi}{3},1) + z_x(\tfrac{\pi}{3},1)(x - \tfrac{\pi}{3}) + z_y(\tfrac{\pi}{3},1)(y-1)$$

$$= 2 + (-2\sqrt{3})(x - \tfrac{\pi}{3}) - \tfrac{2}{3}\sqrt{3}\pi(y-1)$$

$$= 2 - 2\sqrt{3}x + \tfrac{2\sqrt{3}}{3}\pi - \tfrac{2}{3}\sqrt{3}\pi y + \tfrac{2\sqrt{3}}{3}\pi = 2 + \tfrac{4\sqrt{3}}{3}\pi - 2\sqrt{3}x - \tfrac{2\sqrt{3}}{3}\pi y$$

is a plane tangent to $z = 4\cos xy$ at $(\frac{\pi}{3}, 1, 2)$

32. We want to find equations for the plane tangent to $z = F(x,y) = x^2 - 6x + y^3$ that are parallel to $4x - 12y + z = 7$. This means that the plane must have normal vector parallel to that of $4x - 12y + z = 7$.

$4\hat{i} - 12\hat{j} + \hat{k}$ is a normal vector of $4x - 12y + z = 7$. Thus, we must find a point $(x_0, y_0, F(x_0,y_0))$ that lies on $z = F(x,y)$ such that

$$(-F_x(x_0,y_0), -F_y(x_0,y_0), 1) = (4, -12, 1)a \quad \text{for } a \in \mathbb{R}.$$

Since $F_x(x,y) = 2x - 6$, $F_y(x,y) = 3y^2$, we have

$$\begin{cases} 6-2x_0 = 4a \\ -3y_0^2 = -12a \\ 1 = a \end{cases} \Rightarrow \begin{cases} x_0 = \frac{6-4a}{2} = 1 \\ y_0 = \sqrt{4a} = \pm 2 \\ a = 1 \end{cases}$$

Hence the desired points are $\begin{cases} (x_0, y_{0,1}, F(x_0, y_{0,1})) = (1, 2, 3) \text{ and} \\ (x_0, y_{0,2}, F(x_0, y_{0,2})) = (1, -2, -13) \end{cases}$.

And the desired planes are

$$z = F(x_0, y_{0,1}) + F_x(x_0, y_{0,1})(x - x_0) + F_y(x_0, y_{0,1})(y - y_{0,1})$$

$$= 3 - 4(x-1) + 12(y-2) = -17 - 4x + 12y$$

and

$$z = F(x_0, y_{0,2}) + F_x(x_0, y_{0,2})(x - x_0) + F_y(x_0, y_{0,2})(y - y_{0,2})$$

$$= -13 - 4(x-1) + 12(y+2) = 15 - 4x + 12y.$$