HW 12

[5.2 : 29]

29. $f(x,y) = \begin{cases} 1, & x \in \mathbb{Q}; \\ 0, & x \notin \mathbb{Q},\ y \le 1; \\ 2, & x \notin \mathbb{Q},\ y > 1. \end{cases}$

(a) For x rational: $\int_0^2 f(x,y)\,dy = \int_0^2 dy = y\Big|_0^2 = 2$

If x is irrational: $\int_0^2 f(x,y)\,dy = \int_0^1 f(x,y)\,dy + \int_1^2 f(x,y)\,dy$

$$= 0 + \int_1^2 2\,dy = 2\int_1^2 dy = 2y\Big|_1^2$$

$$= 4-2 = 2$$

Thus no matter x is rational or irrational, $\int_0^2 f(x,y)\,dy = 2$

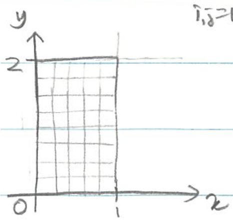
(b) $\int_0^1 \int_0^2 f(x,y)\,dy\,dx = \int_0^1 2\,dx = 2\int_0^1 dx = 2x\Big|_0^1 = 2$

(c) $R = [0,1]\times[0,2]$, $R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j] \subset R$ $(i,j = 1,\dots,n)$

$S = \sum_{i,j=1}^{n} f(\vec{c}_{ij})\,\Delta A_{ij}$ where $\vec{c}_{ij} \in R_{ij} \cap \mathbb{Q}\times\mathbb{R}$, $\Delta A_{ij} = \Delta x_i \Delta y_j$.

$= \sum_{i,j=1}^{n} f(x_i, y_j)\,\Delta x_i \Delta y_j$ where $x_i \in \mathbb{Q}\cap[0,1]$.

$= \sum_{i,j=1}^{n} \Delta x_i \Delta y_j \xrightarrow[\Delta x_i, \Delta y_j \to 0]{}$ Area of $R = 2$



(d) $R = [0,1]\times[0,2]$, $R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j] \subset R$ $(i,j = 1,\dots,n)$

$S = \sum_{i,j=1}^{n} f(x_i^*, y_j^*)\,\Delta x_i \Delta y_j$ where $(x_i^*, y_j^*) \in \Big(\big(\mathbb{Q}\times[0,1]\big) \cup \big((\mathbb{R}\setminus\mathbb{Q})\times(1,2]\big)\Big) \cap R_{ij}$

$= \sum_{i=1}^{n}\sum_{j=1}^{j_0} f(x_i^*, y_j^*)\,\Delta x_i \Delta y_j + \sum_{i=1}^{n}\sum_{j=j_0+1}^{n} f(x_i^*, y_j^*)\,\Delta x_i \Delta y_j$ where $y_{j_0}^* \le 1$ and $y_{j_0+1}^* > 1$.

$= \sum_{i=1}^{n}\sum_{j=1}^{j_0} \Delta x_i \Delta y_j + 2\sum_{i=1}^{n}\sum_{j=j_0+1}^{n} \Delta x_i \Delta y_j \xrightarrow[\Delta x_i, \Delta y_j \to 0]{}$ Area of $[0,1]\times[0,1]$ + 2 × Area of $[0,1]\times[1,2]$

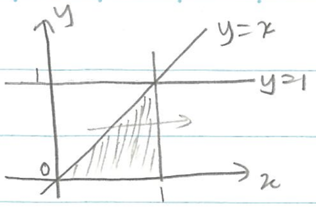
$= 1 + 2\cdot 1 = 1+2 = 3.$

(e) From (c) and (d), $\lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\vec{c}_{ij})\,\Delta x_i \Delta y_j$ doesn't exist since we have two different limits for two different partitions. Hence, the double integral $\iint_R f\,dA$ does not exist.

(But we showed in (a) that the iterated integral $\int_0^1\int_0^2 f\,dy\,dx$ exists.)

[5.3 : 2, 4, 6, 16]

2.



$$\int_0^1\int_0^x (2-x-y)\,dy\,dx = \int_0^1\int_y^1 (2-x-y)\,dx\,dy$$

$$\int_0^1\int_0^x (2-x-y)\,dy\,dx = \int_0^1 \left(2y - xy - \tfrac{1}{2}y^2\right)\Big|_0^x dx = \int_0^1 \left(2x - x^2 - \tfrac{1}{2}x^2\right) dx$$

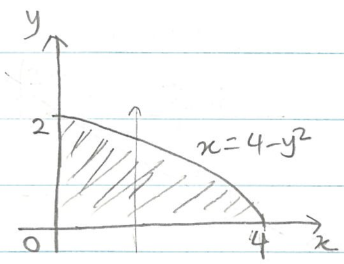
$$= \int_0^1 \left(2x - \tfrac{3}{2}x^2\right) dx = x^2 - \tfrac{1}{2}x^3 \Big|_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2} \quad //$$

$$\int_0^1\int_y^1 (2-x-y)\,dx\,dy = \int_0^1 \left(2x - \tfrac{1}{2}x^2 - xy\right)\Big|_y^1 dy = \int_0^1 \left(2 - \tfrac{1}{2} - y - 2y + \tfrac{1}{2}y^2 + y^2\right) dy$$

$$= \int_0^1 \left(\tfrac{3}{2} - 3y + \tfrac{3}{2}y^2\right) dy = \tfrac{3}{2}y - \tfrac{3}{2}y^2 + \tfrac{1}{2}y^3 \Big|_0^1$$

$$= \tfrac{3}{2} - \tfrac{3}{2} + \tfrac{1}{2} = \tfrac{1}{2} \quad //$$

4.



$$\int_0^2\int_0^{4-y^2} x\,dx\,dy = \int_0^4\int_0^{\sqrt{4-x}} x\,dy\,dx$$

$$\int_0^2\int_0^{4-y^2} x\,dx\,dy = \int_0^2 \tfrac{1}{2}x^2 \Big|_0^{4-y^2} dy = \int_0^2 \tfrac{1}{2}(4-y^2)^2\,dy = \tfrac{1}{2}\int_0^2 (16 - 8y^2 + y^4)\,dy$$

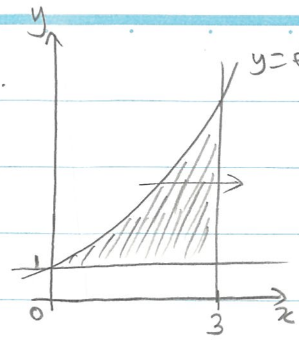
$$= \tfrac{1}{2}\left(16y - \tfrac{8}{3}y^3 + \tfrac{1}{5}y^5\right)\Big|_0^2 = \tfrac{1}{2}\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 16 - \tfrac{32}{3} + \tfrac{16}{5} = \tfrac{128}{15} \quad //$$

$$\int_0^4\int_0^{\sqrt{4-x}} x\,dy\,dx = \int_0^4 xy \Big|_0^{\sqrt{4-x}} dx = \int_0^4 x\sqrt{4-x}\,dx = -\tfrac{2}{3}x(4-x)^{3/2}\Big|_0^4 + \int_0^4 \tfrac{2}{3}(4-x)^{3/2}dx$$

$$= -\tfrac{2}{3}4(4-4)^{3/2} - 0 + \tfrac{2}{3}\frac{(4-x)^{5/2}}{-5/2}\Big|_0^4$$

$$= 0 - \tfrac{4}{15}\left((4-4)^{5/2} - 4^{5/2}\right) = \frac{4^{7/2}}{15} = \frac{2^7}{15} = \frac{128}{15} \quad //$$

6.



$$\int_0^3\int_1^{e^x} 2\,dy\,dx = \int_1^{e^3}\int_{\ln y}^3 2\,dx\,dy$$

$$\int_0^3\int_1^{e^x} 2\,dy\,dx = 2\int_0^3 y\Big|_1^{e^x} dx = 2\int_0^3 (e^x-1)\,dx$$

$$= 2\left(e^x - x\right)\Big|_0^3 = 2\left(e^3-3-1\right) = 2(e^3-4)$$

$$= 2e^3-8 \,/\!/$$

$$\int_1^{e^3}\int_{\ln y}^3 2\,dx\,dy = 2\int_1^{e^3} x\Big|_{\ln y}^3 dy = 2\int_1^{e^3}(3-\ln y)\,dy$$

$$= 2\left(3y-(y\ln y - y)\right)\Big|_1^{e^3} = 2\left(\cancel{3e^3} - \cancel{e^3\cdot 3} + e^3 - 3 - 1\right)$$

$$= 2(e^3-4) = 2e^3-8 \,/\!/$$

16. $$\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy = \int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\,dy\,dx = \int_0^{\pi}\frac{\sin x}{x}\, y\Big|_0^x dx$$

$$= \int_0^{\pi}\frac{\sin x}{x}\cdot x\,dx = \int_0^{\pi}\sin x\,dx = -\cos x\Big|_0^{\pi}$$

$$= -(-1-1) = -(-2) = 2 \,/\!/$$