HW 11

[4.2 : 32, 36, 42, 44]

32. $T(x,y) = 2x^2 + y^2 - y + 3$ $\qquad R: x^2+y^2 \le 1$

$\partial R: x^2+y^2 = 1$

We first find the critical points of T: since T is defined everywhere on R, we only need to consider the points (x_0, y_0) such that $DT(x_0,y_0) = \vec{0}$.

$DT(x_0,y_0) = \vec{0}$

$$(4x_0 \quad 2y_0 - 1) = \vec{0} \iff \begin{cases} 4x_0 = 0 \\ 2y_0 - 1 = 0 \end{cases} \iff \begin{cases} x_0 = 0 \\ y_0 = 1/2 \end{cases}$$

Thus $(0, 1/2)$ is the only critical point of T.

Note that $(0, 1/2)$ lies in the interior of R, since $0 + \frac{1}{4} \le 1$.

The temperature at $(0, 1/2)$ is $T(0, 1/2) = 0 + \frac{1}{4} - \frac{1}{2} + 3 = 11/4$.

Now we check the temperature at the boundary of R, ∂R:

$\partial R = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$.

The temperature on ∂R is given by 1. $T(x, \sqrt{1-x^2})$ on the upper half of xy-plane
2. $T(x, -\sqrt{1-x^2})$ on the lower half of the xy-plane

1. The critical points of $T(x, \sqrt{1-x^2}) = 2x^2 + (1-x^2) - \sqrt{1-x^2} + 3 = x^2 - \sqrt{1-x^2} + 4$ for $x \in [-1,1]$:

Let $f_1(x) := T(x, \sqrt{1-x^2}) = x^2 - \sqrt{1-x^2} + 4$, then

$$f_1'(x) = 2x - \frac{-2x}{\sqrt{1-x^2}} \cdot \frac{1}{2} = 2x + \frac{x}{\sqrt{1-x^2}}$$

$$f_1'(x_0) = 0 \iff 2x_0 + \frac{x_0}{\sqrt{1-x_0^2}} = 0 \iff x_0\left(2 + \frac{1}{\sqrt{1-x_0^2}}\right) = 0$$

The only solution to the above equation for $x_0 \in \mathbb{R}$ is $x_0 = 0$ and $T(0,1) = f_1(0) = 3$.

The temperature at $(1,0)$ and $(-1,0)$ is $f_1(1) = f_1(-1) = 5$.

2. The critical points of $T(x, -\sqrt{1-x^2}) = 2x^2 + (1-x^2) + \sqrt{1-x^2} + 3 = x^2 + \sqrt{1-x^2} + 4$ for $x \in [-1,1]$:

Let $f_2(x) := T(x, -\sqrt{1-x^2}) = x^2 + \sqrt{1-x^2} + 4$, then

$$f_2'(x) = 2x - \frac{2x}{\sqrt{1-x^2}} \cdot \frac{1}{2} = 2x - \frac{x}{\sqrt{1-x^2}}$$

$$f_2'(x_0) = 0 \iff 2x_0 - \frac{x_0}{\sqrt{1-x_0^2}} = 0 \iff x_0\left(2 - \frac{1}{\sqrt{1-x_0^2}}\right) = 0 \iff \begin{cases} x_0 = 0, \text{ or} \\ x_0 = \pm\frac{1}{2}\sqrt{3} \end{cases}$$

$T(0,-1) = f_2(0) = 0+1+4 = 5$

$T(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2}) = f_2(\pm\frac{1}{2}\sqrt{3}) = \frac{3}{4} + \frac{1}{2} + 4 = \frac{21}{4}$

The temperature at $(1,0)$ and $(-1,0)$ is $f_2(1) = f_2(-1) = 5$.

Concluding the above calculations, the hottest points on the plate are $(\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and $(-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and the temperature there is $\frac{21}{4}$. Meanwhile the coldest point on the plate is $(0, \frac{1}{2})$ and the temperature there is $\frac{11}{4}$. //

36. $f(x,y) = x^2y^2$.

(a) Check the critical point: $Df(x,y) = (2xy^2 \quad 2x^2y) = (0 \quad 0)$

$\Leftrightarrow \quad x = y = 0$.

Since the function is defined everywhere on $\mathbb{R}^2$, $(0,0)$ is its only critical pt.

$f_x = 2xy^2$, $f_{xx} = 2y^2$, $f_y = 2x^2y$, $f_{yy} = 2x^2$

$f_{xy} = f_{yx} = 4xy$

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $\det Hf(0,0) = 0$, therefore the critical point $(0,0)$ is degenerate.

The Hessian fails to provide any information about the nature of the critical point $(0,0)$. //

(b) For $(x,y) \neq (0,0)$, $f(x,y) = x^2y^2 \geq 0$.

When $x=0$ or $y=0$, $f(x,y) = 0$ and when $x \neq 0$ and $y \neq 0$, $f(x,y) > 0$.

Thus, f has a local minimum at $(0,0)$ and it also gives a global minimum.

42. $f(x,y) = e^{x^2+5y^2}$.

(a) Since f is defined everywhere on $\mathbb{R}^2$, the points (x_0,y_0) such that $Df(x_0,y_0)=\vec{0}$ are the only possible critical points of f.

$Df(x_0,y_0)=\vec{0}$

$$\left(2x_0e^{x_0^2+5y_0^2} \quad 10y_0e^{x_0^2+5y_0^2}\right)=\vec{0} \iff \begin{cases} 2x_0e^{x_0^2+5y_0^2}=0 \\ 10y_0e^{x_0^2+5y_0^2}=0 \end{cases} \iff \begin{cases} x_0=0 \\ y_0=0. \end{cases}$$

Thus $(0,0)$ is the only critical point of f.

$f_x = 2xe^{x^2+5y^2}$, $f_{xx} = 2e^{x^2+5y^2}+4x^2e^{x^2+5y^2}$, $f_{xy}=20xye^{x^2+5y^2}$

$f_y = 10ye^{x^2+5y^2}$, $f_{yy} = 10e^{x^2+5y^2}+100y^2e^{x^2+5y^2}$, $f_{yx}=20xye^{x^2+5y^2}$

$$Hf(0,0)=\begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0)\end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 0 & 10\end{pmatrix}, \quad \det Hf(0,0)=20\neq 0.$$

Since $f_{xx}(0,0)=2>0$ and $\det Hf(0,0)=20>0$, f has a local minimum at $(0,0)$. //

(b) $f(0,0)=1$ and for $(x,y)\neq(0,0)$, $x^2+5y^2>0$.

e^x is a strictly increasing function of x, thus $e^{x^2+5y^2}>1$ for $(x,y)\neq(0,0)$.

Therefore, f has a global minimum at $(0,0)$ and its global minimum is 1 //

44. $f(x,y) = x^3+y^3-3xy+7$.

(a) Since f is defined everywhere on $\mathbb{R}^2$, the points (x_0,y_0) such that $Df(x_0,y_0)=\vec{0}$ are the only critical points of f.

$$Df(x_0,y_0)=\vec{0} \iff \left(3x_0^2-3y_0 \quad 3y_0^2-3x_0\right)=(0 \quad 0) \iff \begin{cases} 3x_0^2-3y_0=0 \\ 3y_0^2-3x_0=0 \end{cases}$$

$$\iff \begin{cases} x_0^2-y_0=0 \\ y_0^2-x_0=0 \end{cases}$$

$$\begin{cases} x_0^2-y_0=0 \\ y_0^2-x_0=0 \end{cases} \iff \begin{cases} x_0^2y_0-y_0^2=0 & \cdots ① \\ y_0^2-x_0=0 & \cdots ② \end{cases}$$

$① + ②$: $x_0(x_0y_0-1)=0$

$x_0=0$ or $x_0y_0=1$

$x_0 = 0$ gives $y_0 = 0$ and $x_0 y_0 = 1$ gives $(1,1)$ or $(-1,-1)$.

However, $(1,1)$ satisfy both ① and ② but $(-1,-1)$ do not satisfy both ① and ②.

Hence, the only solutions to ① and ② are $(0,0)$ and $(1,1)$.

We can also check this as follows:

$$\begin{cases} x_0^2 - y_0 = 0 \\ y_0^2 - x_0 = 0 \end{cases} \iff y_0^4 - y_0 = 0 \iff y_0(y_0^3 - 1) = 0$$

Hence $y_0 = 0$ or $y_0^3 = 1$ which implies $y_0 = 1$.

Inserting $y_0 = 0$ into the first two equations we have $x_0 = 0$ and $y_0 = 1$ gives $x_0 = 1$

$(0,0)$ and $(1,1)$ are the only critical points of f.

$f_x = 3x^2 - 3y$, $f_{xx} = 6x$, $f_{xy} = -3$

$f_y = 3y^2 - 3x$, $f_{yy} = 6y$, $f_{yx} = -3$.

$$Hf(0,0) = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}, \quad \det Hf(0,0) = -9.$$

$$Hf(1,1) = \begin{pmatrix} f_{xx}(1,1) & f_{xy}(1,1) \\ f_{yx}(1,1) & f_{yy}(1,1) \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}, \quad \det Hf(1,1) = 27.$$

Since $\det Hf(0,0) \neq 0$ but $f_{xx}(0,0) = 0$, f has a saddle point at $(0,0)$.

Since $f_{xx}(1,1) > 0$ and $\det Hf(1,1) > 0$, f has a local minimum at $(1,1)$.

(b) $f(0,0) = 7$, $f(1,1) = 1 + 1 - 3 + 7 = 6$.

Along the line $x = 0$, $f(0,y) = y^3 + 7$ is a monotone increasing function of y (and similarly along the line $y = 0$, $f(x,0) = x^3 + 7$ is a monotone increasing function of x).

Furthermore, as $y \to +\infty$, $f(y,0) \to +\infty$ and as $y \to -\infty$, $f(y,0) \to -\infty$ (similarly $f(x,0) \to \pm\infty$ as $x \to \pm\infty$).

Thus, f has no global extrema, neither maximum nor minimum.

[5-1: 2, 4, 6, 8, 10, 12]

2. $\int_0^{\pi}\int_1^2 y\sin x\,dy\,dx = \int_0^{\pi} \sin x \left(\frac{1}{2}y^2\Big|_1^2\right)dx = \int_0^{\pi}(\sin x)\left(\frac{4}{2}-\frac{1}{2}\right)dx$

$= \frac{3}{2}\int_0^{\pi}\sin x\,dx = -\frac{3}{2}\cos x\Big|_0^{\pi} = -\frac{3}{2}(-1-1) = 3$ //

4. $\int_0^{\pi/2}\int_0^1 e^x\cos y\,dx\,dy = \int_0^{\pi/2}\cos y\left[e^x\right]_0^1 dy = \int_0^{\pi/2}\cos y\ (e-1)\,dy$

$= (e-1)\sin y\Big|_0^{\pi/2} = (e-1)(1-0) = e-1$ //

6. $\int_1^9\int_1^e \frac{\ln\sqrt{x}}{xy}\,dx\,dy = \int_1^9 \frac{1}{y}\,dy \int_1^e \frac{\frac{1}{2}\ln x}{x}\,dx$

$= \left(\ln|y|\Big|_1^9\right)\cdot\frac{1}{2}\int_1^e \ln x\, d(\ln x) = \frac{1}{2}\ln 9\left(\frac{1}{2}(\ln x)^2\right)\Big|_1^e$

$= \frac{1}{2}\ln 3^2 \cdot \frac{1}{2}(\ln e)^2 = \frac{1}{2}\ln 3$ //

8. $\int_1^2\int_0^3 (x+3y+1)\,dx\,dy = \int_1^2\left[\frac{1}{2}x^2+3xy+x\right]_0^3 dy$

$= \int_1^2\left(\frac{9}{2}+9y+3\right)dy = \int_1^2\left(9y+\frac{15}{2}\right)dy$

$= \frac{9}{2}y^2+\frac{15}{2}y\Big|_1^2 = 18+15-\frac{9}{2}-\frac{15}{2}$

$= 33-12 = 21$ //

10. $\int_0^2\int_1^3 2\,dx\,dy = 2\,y\Big|_0^2\ x\Big|_1^3 = 2(2-0)(3-1) = 8$ //

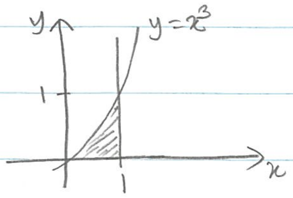
The above integral represents the volume of the region bounded on the top by the plane $z=2$, on the bottom by the xy-plane $(z=0)$, and on the sides by the planes $x=1$, $x=3$, $y=0$, $y=2$. //

12. $\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \sin x\cos y\,dx\,dy = \sin y\Big|_{-\pi/2}^{\pi/2}\cdot(-\cos x)\Big|_0^{\pi} = (1-(-1))(-(-1)+1) = 4$ //

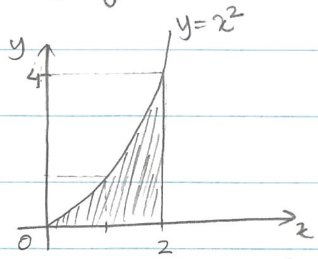
The above integral represents the volume of the region bounded on the top by the plane $z=\sin x\cos y$, on the bottom by the xy-plane $(z=0)$, and on the sides by the planes $x=0$, $x=\pi$, $y=-\frac{\pi}{2}$, $y=\frac{\pi}{2}$ //

[5.2 : 2, 4, 12].

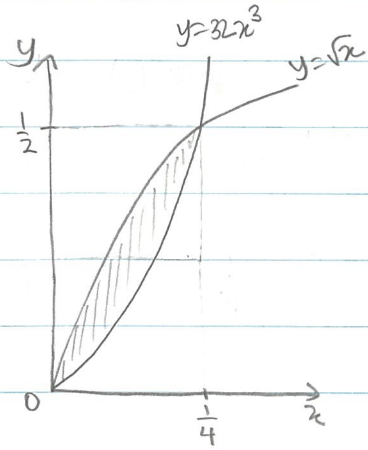
2. $\displaystyle\int_0^1\int_0^{x^3} 3\,dy\,dx = 3\int_0^1 y\Big|_0^{x^3} dx = 3\int_0^1 x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$ //



4. $\displaystyle\int_0^2\int_0^{x^2} y\,dy\,dx = \int_0^2 \frac{1}{2}y^2\Big|_0^{x^2} dx = \int_0^2 \frac{1}{2}x^4\,dx = \frac{1}{10}x^5\Big|_0^2 = \frac{32}{10} = \frac{16}{5}$ //



12.



$32x^3 = \sqrt{x}$

$(32)^2 x^6 = x$

$32^2 x^6 - x = 0$

$x(32^2 x^5 - 1) = 0$

$x = 0$ or $x^5 = \frac{1}{32^2}$

$x = \frac{1}{4}$

$$\int_0^{\frac{1}{4}}\int_{32x^3}^{\sqrt{x}} 3xy\,dy\,dx$$

$$= 3\int_0^{1/4} x\cdot\frac{1}{2}y^2\Big|_{32x^3}^{\sqrt{x}} dx$$

$$= \frac{3}{2}\int_0^{1/4} x\,(x - 32^2x^6)\,dx$$

$$= \frac{3}{2}\int_0^{1/4}(x^2 - 32^2x^7)\,dx$$

$$= \frac{3}{2}\left(\frac{1}{3}x^3 - \frac{32^2}{8}x^8\right)\Big|_0^{1/4}$$

$$= \frac{3}{2}\left(\frac{1}{3}\frac{1}{2^6} - \frac{2^{10}}{2^3}\frac{1}{2^{16}}\right)$$

$$= \frac{3}{2}\left(\frac{1}{3}\frac{1}{2^6} - \frac{1}{2^9}\right) = \frac{1}{2^7} - \frac{3}{2^{10}} = \frac{5}{2^{10}}$$

$$= \frac{1}{128} - \frac{3}{1024} = \frac{5}{1024}$$ //

The volume is $5/1024$ unit volume.