HW 6

[4.1 : 27, 28, 30]

27. (a) $f(x,y) = x^2y^3$, $\vec{a} = (7,2)$, $\vec{h} = (0.07, -0.02)$

$df(x,y) = 2xy^3\,dx + 3x^2y^2\,dy$

$$df\left(\begin{pmatrix}7\\2\end{pmatrix}, \begin{pmatrix}0.07\\-0.02\end{pmatrix}\right) = 2\cdot7\cdot2^3(0.07) + 3\cdot7^2\cdot2^2(-0.02)$$

$$= 7.84 - 11.76 = -3.92$$

Since $df\left(\begin{pmatrix}7\\2\end{pmatrix}, \begin{pmatrix}0.07\\-0.02\end{pmatrix}\right) \approx \Delta f = f(7.07, 1.98) - f(7,2)$

$$= (7.07)^2(1.98)^3 - 392$$

$$(7.07)^2(1.98)^3 \approx 392 - 3.92 = 388.08 \;//$$

(b) $f(x,y,z) = \frac{1}{\sqrt{xyz}}$, $\vec{a} = (4,2,2)$, $\vec{h} = (0.1, -0.04, 0.05)$

As in part (a), $f(4.1, 1.96, 2.05) \approx f(4,2,2) + df\left(\begin{pmatrix}4\\2\\2\end{pmatrix}, \begin{pmatrix}0.1\\-0.04\\0.05\end{pmatrix}\right)$

$f(4,2,2) = \frac{1}{\sqrt{4\cdot2\cdot2}} = \frac{1}{4}$ (consider only for $f(x,y,z) \geq 0$).

$f_x = \frac{-\frac{1}{2}yz}{(xyz)^{3/2}}$, $f_y = \frac{-\frac{1}{2}xz}{(xyz)^{3/2}}$, $f_z = \frac{-\frac{1}{2}xy}{(xyz)^{3/2}}$

$$df\left(\begin{pmatrix}4\\2\\2\end{pmatrix}, \begin{pmatrix}0.1\\-0.04\\0.05\end{pmatrix}\right) = f_x(4,2,2)\cdot 0.1 + f_y(4,2,2)\cdot(-0.04) + f_z(4,2,2)\cdot 0.05$$

$$= -\frac{1}{\cancel{2}}\frac{4^{\cancel{2}}}{16\cdot4_2}(0.1) - \frac{1}{\cancel{2}}\frac{\cancel{8}^4}{16\cdot4}(-0.04) - \frac{1}{\cancel{2}}\frac{\cancel{8}^4}{16\cdot4}(0.05)$$

$$= -0.003125 + 0.0025 - 0.003125 = -0.00375$$

$\therefore \frac{1}{\sqrt{(4.1)(1.96)(2.05)}} \approx \frac{1}{4} - 0.00375 = 0.25 - 0.00375 = 0.24625 \;//$

(If we let $f(x,y,z)$ can be both $f(x,y,z) \geq 0$ and $f(x,y,z) \leq 0$,

then $\frac{1}{\sqrt{(4.1)(1.96)(2.05)}} \approx \pm 0.24625 \;//$ (this can be checked easily!))

(c) $f(x,y,z) = x\cos((\pi-y)z)$, $\vec{a} = (1,0,0)$, $\vec{h} = (0.1, 0.03, 0.12)$

As in part (a) and (b), $f(1.1, 0.03, 0.12) \approx f(1,0,0) + df\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0.1\\0.03\\0.12\end{pmatrix}\right)$.

$f(1,0,0) = 1\cdot\cos(\pi\cdot 0) = 1.1 = 1$

$f_x = \cos((\pi-y)z)$, $f_y = xz\sin((\pi-y)z)$, $f_z = -x(\pi-y)\sin((\pi-y)z)$

$$df\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0.1\\0.03\\0.12\end{pmatrix}\right) = f_x(1,0,0)\cdot 0.1 + f_y(1,0,0)\cdot 0.03 + f_z(1,0,0)\cdot 0.12$$

$$= \cos(\pi.0)\cdot 0.1 + 0\cdot 0.03 - 1\cdot\pi\cdot\sin 0\cdot 0.12$$

$$= 0.1 + 0 - 0 = 0.1$$

$\therefore (1.1)\cos((\pi - 0.03)(0.12)) \approx 1 + 0.1 = 1.1$ //

28. $g(x,y,z) = x^3 - 2xy + x^2z + 7z$, $\vec{a} = (1,-2,1)$, $\vec{h} = (\Delta x, \Delta y, \Delta z)$

$\Delta g \approx dg = dg(\vec{a},\vec{h}) = g_x(\vec{a})\Delta x + g_y(\vec{a})\Delta y + g_z(\vec{a})\Delta z$

$g_x = 3x^2 - 2y + 2xz$, $g_x(\vec{a}) = 3 - 2(-2) + 2(1)(1) = 9$.

$g_y = -2x$, $g_y(\vec{a}) = -2$

$g_z = x^2+7$, $g_z(\vec{a}) = 1 + 7 = 8$

$\therefore \Delta g \approx 9\Delta x - 2\Delta y + 8\Delta z$.

Hence g is most sensitive to changes in x. //

30. $(r,h) = (2,3)$, $\vec{e} = (\pm 0.1, \pm 0.05)$

$V(r,h) = \pi r^2 h$, $S(r,h) = 2\pi rh + 2\pi r^2 = 2\pi r(h+r)$.

(a) $\Delta V \approx dV = V_r(2,3)\Delta r + V_h(2,3)\Delta h = 2\pi rh\Big|_{(r,h)=(2,3)}(\pm 0.1) + \pi r^2\Big|_{(r,h)=(2,3)}(\pm 0.05)$

$$= 12\pi(\pm 0.1) + 4\pi(\pm 0.05) = (\pm 1.2\pi) + (\pm 0.2\pi) = \pm 1.4\pi$$ //

(b) $\Delta S \approx dS = S_r(2,3)\Delta r + S_h(2,3)\Delta h = 2\pi(h+2r)\Big|_{(r,h)=(2,3)}(\pm 0.1) + 2\pi r\Big|_{(r,h)=(2,3)}(\pm 0.05)$

$$= 14\pi(\pm 0.1) + 4\pi(\pm 0.05) = (\pm 1.4\pi) + (\pm 0.2\pi) = \pm 1.6\pi$$ //

[4.2 : 2, 4, 8, 14, 22]

2. $g(x,y) = x^2 - 2y^2 + 2x + 3$.

(a) $Dg(x,y) = \begin{pmatrix} 2x+2 & -4y \end{pmatrix}$

Since polynomials are defined everywhere on $\mathbb{R}$, the critical points of g are all given by the points where $Dg = \vec{0}$, that is $\begin{cases} 2x+2=0 \\ -4y=0 \end{cases}$ which gives

$x=-1$ and $y=0$. Thus $(-1,0)$ is the only critical point of g. //

(b) $\Delta g = g(-1+h, 0+k) - g(-1,0)$

$= (-1+h)^2 - 2k^2 + 2(-1+h) + 3 - \left((-1)^2 - 2 \cdot 0 + 2(-1) + 3 \right)$

$= 1 - \cancel{2h} + h^2 - 2k^2 - 2 + \cancel{2h} + 3 - (1 - 2 + 3)$

$= h^2 - 2k^2 + \cancel{2} - \cancel{2} = h^2 - 2k^2$.

when $h > \sqrt{2}k$, we have $\Delta g > 0$, but when $h < \sqrt{2}k$, we have $\Delta g < 0$.

For example, when $h = 0.1$, $k=0$, $\Delta g = 0.01 > 0$, but when $h = 0.1$, $k = 0.1$

$\Delta g = 0.01 - 0.02 = -0.01 < 0$

Thus $(-1,0)$ is a saddle point of g.

(c) $g_x = 2x+2$, $g_{xx} = 2$, $g_{xy} = g_{yx} = 0$.

$g_y = -4y$, $g_{yy} = -4$

$$Hg(-1,0) = \begin{pmatrix} g_{xx}(-1,0) & g_{xy}(-1,0) \\ g_{yx}(-1,0) & g_{yy}(-1,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

$g_{xx}(-1,0) = 2 > 0$ and $|Hg(-1,0)| = -8 < 0$, thus g has a saddle point at $(-1,0)$ //

4. $f(x,y) = \ln(x^2+y^2+1)$

$f_x = \dfrac{2x}{x^2+y^2+1}$, $f_{xx} = \dfrac{2(x^2+y^2+1) - 2x(2x)}{(x^2+y^2+1)^2} = \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$

$f_y = \dfrac{2y}{x^2+y^2+1}$, $f_{yy} = \dfrac{2(x^2+y^2+1) - 2y(2y)}{(x^2+y^2+1)^2} = \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$

$f_{xy} = f_{yx} = \dfrac{-2x(2y)}{(x^2+y^2+1)^2} = \dfrac{-4xy}{(x^2+y^2+1)^2}$

$$Df(x,y) = \begin{pmatrix} \frac{2x}{x^2+y^2+1} & \frac{2y}{x^2+y^2+1} \end{pmatrix} = (0\ \ 0)$$

$$\Leftrightarrow\ x=0,\ y=0$$

Since f_x and f_y are rational functions defined everywhere on $\mathbb{R}^2$, $(0,0)$ is the only critical point of f. //

$$Hf(0,0) = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$f_{xx}(0,0) = 2 > 0$ and $|Hf(0,0)| = 4 > 0$, thus f has a local minimum at $(0,0)$ //

8. $f(x,y) = e^x \sin y$

$f_x = e^x \sin y$, $f_{xx} = e^x \sin y$, $f_{xy} = f_{yx} = e^x \cos y$

$f_y = e^x \cos y$, $f_{yy} = -e^x \sin y$

$$Df(x,y) = (e^x \sin y\ \ \ e^x \cos y) = (0\ \ 0)$$

$$\Leftrightarrow (\sin y\ \ \cos y) = (0\ \ 0) \quad (\text{since } e^x > 0 \text{ for all } x \in \mathbb{R}).$$

But this such point does not exist.

Furthermore, f_x and f_y are defined everywhere on $\mathbb{R}^2$. Thus f does not have any critical points.

14. $f(x,y) = \cos x \sin y$

$f_x = -\sin x \sin y$, $f_{xx} = -\cos x \sin y$, $f_{xy} = f_{yx} = -\sin x \cos y$

$f_y = \cos x \cos y$, $f_{yy} = -\cos x \sin y$

$$Df(x,y) = (-\sin x \sin y\ \ \ \cos x \cos y) = (0\ \ 0)$$

$$\Leftrightarrow \begin{cases} (x,y) = (m\pi, (2n+1)\frac{\pi}{2}) \text{ for any } m,n \in \mathbb{Z}, \text{ or} \\ (x,y) = ((2m+1)\frac{\pi}{2}, n\pi) \text{ for any } m,n \in \mathbb{Z}. \end{cases}$$

$$\begin{cases} Hf(m\pi, (2n+1)\frac{\pi}{2}) = \begin{pmatrix} -\cos(m\pi)\sin((2n+1)\frac{\pi}{2}) & 0 \\ 0 & -\cos(m\pi)\sin((2n+1)\frac{\pi}{2}) \end{pmatrix}, & m,n \in \mathbb{Z} \\ Hf((2m+1)\frac{\pi}{2}, n\pi) = \begin{pmatrix} 0 & -\sin((2m+1)\frac{\pi}{2})\cos(n\pi) \\ -\sin((2m+1)\frac{\pi}{2})\cos(n\pi) & 0 \end{pmatrix}, & m,n \in \mathbb{Z} \end{cases}$$

$$\begin{cases} \det Hf(m\pi, (2n+1)\frac{\pi}{2}) = \cos^2(m\pi)\sin^2((2n+1)\frac{\pi}{2}) = 1 > 0 & \text{for any } m,n \in \mathbb{Z}. \\ \det Hf((2m+1)\frac{\pi}{2}, n\pi) = -\sin^2((2m+1)\frac{\pi}{2})\cos^2(n\pi) = -1 < 0 & \text{for any } m,n \in \mathbb{Z}. \end{cases}$$

Because $f_{xx}(m\pi, (2n+1)\frac{\pi}{2}) = -\cos(m\pi)\sin((2n+1)\frac{\pi}{2}) = \begin{cases} -1, & \text{if } m \text{ and } n \text{ have the same parity (both even or both odd)} \\ 1, & \text{otherwise,} \end{cases}$

thus f has local maximum at $(m\pi, (2n+1)\frac{\pi}{2})$ when m and n have the same parity and local minimum there otherwise.

Furthermore, f has saddle points at $((2m+1)\frac{\pi}{2}, n\pi)$

22. (a) $f(x,y) = kx^2 - 2xy + ky^2$

$f_x = 2kx - 2y$, $f_{xx} = 2k$, $f_{xy} = f_{yx} = -2$.

$f_y = -2x + 2ky$, $f_{yy} = 2k$.

$$\begin{cases} Hf(x,y) = \begin{pmatrix} 2k & -2 \\ -2 & 2k \end{pmatrix} \Rightarrow |Hf(x,y)| = 4k^2 - 4 & \text{for any } (x,y) \in \mathbb{R}^2. \\ f_{xx}(x,y) = 2k & \text{for any } (x,y) \in \mathbb{R}^2. \end{cases}$$

$Df(0,0) = (0\ 0)$

hence $(0,0)$ is a critical point of f.

- If we want f to have a nondegenerate local minimum at $(0,0)$,

$$\begin{cases} 4k^2 - 4 > 0 \\ 2k > 0 \end{cases} \quad \text{must both be satisfied.}$$

$$\Leftrightarrow \begin{cases} 4(k^2-1) > 0 \\ k > 0 \end{cases}$$

Thus letting $k > 1$, f would have a nondegenerate local minimum at $(0,0)$. //

- If we want f to have a nondegenerate local maximum,

$$\begin{cases} 4k^2 - 4 > 0 \\ 2k < 0 \end{cases} \quad \text{must both be satisfied at its critical point.}$$

Thus letting $k < -1$, f would have a nondegenerate local maximum at $(0,0)$. //

(b) $g(x,y,z) = kx^2 + kxz - 2yz - y^2 + \frac{k}{2}z^2$

$g_x = 2kx + kz$, $g_{xx} = 2k$, $g_{xy} = g_{yx} = 0$

$g_y = -2z - 2y$, $g_{yy} = -2$, $g_{yz} = g_{zy} = -2$

$g_z = kx - 2y + kz$, $g_{zz} = k$, $g_{xz} = g_{zx} = k$

$$Hg(x,y,z) = \begin{pmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{pmatrix}$$ for any $(x,y,z) \in \mathbb{R}^3$.

$$\begin{cases} g_{xx} = 2k , \\ \begin{vmatrix} 2k & 0 \\ 0 & -2 \end{vmatrix} = -4k , \\ |Hg(x,y,z)| = 2k(-2k-4) + k(2k) = -4k^2 - 8k + 2k^2 = -2k^2 - 8k , \end{cases}$$

for any $(x,y,z) \in \mathbb{R}^3$

- If we want g to have a nondegenerate local maximum at $(0,0,0)$,

$$\begin{cases} 2k < 0 \\ -4k > 0 \\ -2k^2 - 8k < 0 \end{cases}$$ must all be satisfied.

$Dg(0,0,0) = (0\ 0\ 0)$, hence $(0,0,0)$ is a critical pt. of g

$$\Leftrightarrow \begin{cases} k < 0 \\ -2k(k+4) < 0 \end{cases} \Leftrightarrow \begin{cases} k < 0 \\ k + 4 < 0 \end{cases} \Leftrightarrow k < -4 .$$

Thus letting $k < -4$, g would have a nondegenerate local maximum at $(0,0,0)$. //

- If we want g to have a nondegenerate local minimum,

$$\begin{cases} 2k > 0 \\ -4k > 0 \\ -2k^2 - 8k > 0 \end{cases}$$ must all be satisfied at its critical point.

However, $2k > 0$ and $-4k > 0$ cannot be satisfied at the same time for $k \in \mathbb{R}$.
Thus g cannot have a nondegenerate local minimum at any point $(x,y,z) \in \mathbb{R}^3$ for all $k \in \mathbb{R}$. //