Nagoya University, G30 program Spring 2015

Linear algebra II Instructor : Serge Richard

Final examination

Exercise 1 *Let* $\mathcal{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $\mathrm{L}_{\mathcal{A}}(X) = \mathcal{A}X$ *for any* $X \in \mathbb{R}^3$*. Determine the eigenvalues of* $\mathrm{L}_{\mathcal{A}}$ *and the corresponding eigenspaces. What is the dimension of these eigenspaces ?*

Exercise 2 *(i) Compute the real and the imaginary part of the complex number* $\frac{3+2i}{1+i}$*,*

(ii) Compute the modulus and the argument of the complex numbers i *and* $1 + i\sqrt{3}$*.*

Exercise 3 *For* $k \in \mathbb{R}$*, consider the three vectors*

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 2 \\ 3 \\ k \end{pmatrix}.$$

(i) Find all values of k *for which the three vectors are linearly independent.*

(ii) If the three vectors are linearly independent, do they generate a basis of $\mathbb{R}^3$ *? Justify your answer.*

Exercise 4 *Let* $\mathcal{A}$ *be the matrix given by* $\mathcal{A} = \begin{pmatrix} 0 & 1 & 3 & -2 \\ 1 & 1 & -4 & 3 \\ 1 & 3 & 2 & -1 \end{pmatrix}$ *and consider the linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^4 \to \mathbb{R}^3$ *defined by* $\mathrm{L}_{\mathcal{A}}(X) = \mathcal{A}X$ *for any* $X \in \mathbb{R}^4$*.*

(i) Determine the rank of $\mathcal{A}$ *and the dimension of the range of* $\mathrm{L}_{\mathcal{A}}$*.*

(ii) Deduce the dimension of the kernel of $\mathrm{L}_{\mathcal{A}}$*, and exhibit a basis for the kernel of* $\mathrm{L}_{\mathcal{A}}$*.*

(iii) Compute $\mathrm{L}_{\mathcal{A}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$*, and find the set of all solutions of the equation* $\mathrm{L}_{\mathcal{A}}(X) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$*.*

Exercise 5 *We consider the real vector space* $V := C([0,1])$ *made of continuous real functions on* $[0,1]$ *and endow it with the scalar product*

$$V \times V \ni (f,g) \mapsto \langle f, g \rangle := \int_0^1 f(x)\,g(x)\,\mathrm{d}x \ \in \mathbb{R}.$$

(i) Are the following functions orthogonal with respect to this scalar product: $x \mapsto x$ *and* $x \mapsto x^2$ *?*

(ii) If W *is the subspace of* V *generated by the two functions* $x \mapsto 1$ *and* $x \mapsto x$*, find an orthonormal basis for* W*.*

Exercise 6 *Consider* $\mathcal{A} = {}^t\mathcal{A} \in M_2(\mathbb{R})$ *with its characteristic polynomial* $P_{\mathcal{A}}(\lambda) = (\frac{1}{2} - \lambda)(k - \lambda)$*, where* k *is a real number satisfying* $k > 0$*. Depending on* k*, what can you say about* $\mathcal{A}^n$*, when* n *tends to* ∞ *? Can you also express* $\mathsf{Det}(\mathcal{A}^n)$ *in terms of* k *?*

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Exercise 1

$$P_A(\lambda) = \mathrm{Det}\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{pmatrix} = (2-\lambda)\big((1-\lambda)(4-\lambda)+2\big)$$

$$= (2-\lambda)(\lambda^2 - 5\lambda + 6) = (2-\lambda)(2-\lambda)(3-\lambda).$$

The eigenvalues of L_A are $\underline{2 \text{ and } 3}$.

$\underline{\lambda = 2}$: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -y-z \\ 2y+2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x \text{ arbitrary} \\ y = 0 \\ z = 0 \end{cases}$

Eigenspace : $\underline{\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \mid x \in \mathbb{R} \right\}}$ of $\underline{\text{dim } 1}$.

$\underline{\lambda = 3}$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+y \\ -2y-z \\ 2y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} x = y \\ z = -2y \\ y \text{ arbitrary} \end{cases}$ Eigenspace : $\underline{\left\{ y\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \in \mathbb{R}^3 \mid y \in \mathbb{R} \right\}}$

of $\underline{\text{dimension } 1}$.

Exercise 2

i) $\alpha = \dfrac{3+2i}{1+i} = \dfrac{(3+2i)(1-i)}{(1+i)(1-i)} = \dfrac{3+2i-3i+2}{2} = \dfrac{5}{2} - \dfrac{1}{2}i$

$\Rightarrow \mathrm{Re}(\alpha) = \dfrac{5}{2}$, $\mathrm{Im}(\alpha) = -\dfrac{1}{2}$.

ii) $|i| = 1$ and $\arg(i) = \dfrac{\pi}{2}$.

$|1+i\sqrt{3}| = (1+3)^{1/2} = 2$

$1+i\sqrt{3} = 2(\cos\theta + i\sin\theta)$

$\Rightarrow \theta = 60° = \dfrac{\pi}{3}$.

Exercise 3

1) If $\mathrm{Det}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & k \end{pmatrix} \neq 0$, then the vectors are linearly independent. One has

$$\mathrm{Det}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & k \end{pmatrix} = 1(k+3) - 1(-k+2) + 1(-3-2)$$
$$= k+3+k-2-5 = 2k-4 = 2(k-2).$$

Thus, the vectors are linearly independent if $k \neq 2$.

2) Yes, they generate a basis of $\mathbb{R}^3$ because 3 linearly independent vectors in $\mathbb{R}^3$ generate a space of dimension 3, which can only be $\mathbb{R}^3$.

Exercise 4

1) Only 2 rows are linearly independent because

$$2\,\underset{\uparrow \text{ first row}}{(0\ 1\ 3\ -2)} + 1\,\underset{\uparrow \text{ second row}}{(1\ 1\ -4\ 3)} = \underset{\uparrow \text{ third row}}{(1\ 3\ 2\ -1)}$$

$\Rightarrow$ $\underline{\operatorname{rank} A = 2}$.

2) $\dim \mathbb{R}^4 = \dim \mathrm{Range}(L_A) + \dim \mathrm{Ker}(L_A)$

$\Leftrightarrow \quad 4 = 2 + \dim \mathrm{Ker}(L_A) \quad \Rightarrow \quad \dim \mathrm{Ker}(L_A) = 2.$

$$\begin{pmatrix} 0 & 1 & 3 & -2 \\ 1 & 1 & -4 & 3 \\ 1 & 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 7x_3 - 5x_4 \\ x_2 = -3x_3 + 2x_4 \end{cases}$$

$\Rightarrow$ A basis of $\mathrm{Ker}\, L_A$ is $\left\{ \begin{pmatrix} 7 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

3) $\begin{pmatrix} 0 & 1 & 3 & -2 \\ 1 & 1 & -4 & 3 \\ 1 & 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Thus, all solutions of $L_A(X) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 7 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}.$$

Exercise 5

i) If $f(x) = x$ and $g(x) = x^2$, then

$\langle f, g \rangle = \int_0^1 x\, x^2\, dx = \int_0^1 x^3\, dx = \frac{1}{4} \neq 0$.

These functions are not orthogonal.

ii) Set $f_1(x) = 1 \quad \forall\, x \in [0,1]$.

Then $\int_0^1 1\, x\, dx = \frac{1}{2}$ and thus

$f_2(x) = \frac{1}{c}\left(x - \frac{1}{2}\right)$.

$\|f_2\|^2 = \langle f_2, f_2 \rangle = \frac{1}{c^2} \int_0^1 \left(x - \frac{1}{2}\right)^2 = \frac{1}{c^2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right)$

$= \frac{1}{c^2} \, \frac{4-6+3}{12} = \frac{1}{12} \, \frac{1}{c^2} \equiv 1$

$\Rightarrow c^2 = \frac{1}{12} \quad \Rightarrow \quad c = \frac{1}{\sqrt{12}}$

$f_2(x) := \sqrt{12}\left(x - \frac{1}{2}\right)$ and $\{f_1, f_2\}$ is an orthonormal basis for W.

Exercise 6

One has $\mathcal{A} = B \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & k \end{pmatrix} B^{-1}$ for some invertible matrix B.

If $k > 1$, then $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & k \end{pmatrix}^n \xrightarrow{n \to \infty} \begin{pmatrix} 0 & 0 \\ 0 & \infty \end{pmatrix}$ and thus $\mathcal{A}^n$ has no limit when $n \to \infty$.

If $k = 1$, then $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}^n \xrightarrow{n \to \infty} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

and thus $\mathcal{A}^n \xrightarrow{n \to \infty} B \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} B^{-1}$.

If $k < 1$, then $\mathcal{A}^n \xrightarrow{n \to \infty} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

One has $\operatorname{Det} \mathcal{A} = \frac{1}{2} k$, and

$$\operatorname{Det} \mathcal{A}^n = \left(\frac{k}{2}\right)^n .$$