Chapter 7

Eigenvectors and eigenvalues

7.1 Eigenvalues and eigenvectors

We start with the main definition of this chapter.

Definition 7.1.1. *Let V be a vector space over a field $\mathbb{F}$, and let $\mathrm{L} : V \to V$ be a linear map. An element $\lambda \in \mathbb{F}$ is* an eigenvalue *of* L *if there exists $X \in V$ with $X \neq \mathbf{0}$ such that*

$$\mathrm{L}(X) = \lambda X.$$

In such a case, X is called an eigenvector *or* an eigenfunction *associated with the eigenvalue λ.*

Examples 7.1.2. *(i) Consider $\mathrm{L}_\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathcal{A} = \left(\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right)$. Then one observes that*

$$\mathrm{L}_\mathcal{A} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus, $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ is an eigenvector of $\mathrm{L}_\mathcal{A}$ associated with the eigenvalue 5. Similarly, one can check that $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ is an eigenvector of $\mathrm{L}_\mathcal{A}$ associated with the eigenvalue -1.

(ii) If $\mathcal{A} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mn} \end{pmatrix}$, then E_j is an eigenvector of $\mathrm{L}_\mathcal{A}$ associated with the eigenvalue a_{jj}.

(iii) If $V = C^1(\mathbb{R})$ and if $\mathrm{L} = \frac{\mathrm{d}}{\mathrm{d}x}$, then any $\lambda \in \mathbb{R}$ is an eigenvalue of L *since the function $x \mapsto e^{\lambda x}$ belongs to $C^1(\mathbb{R})$ and satisfies*

$$\big[\mathrm{L}\big(e^{\lambda\cdot}\big)\big](x) = \big(e^{\lambda\cdot}\big)'(x) = \lambda e^{\lambda x}.$$

Thus this function is an eigenvector associated with the eigenvalue λ.

Remark 7.1.3. *An eigenvector is never unique. Indeed, if X is an eigenvector associated with the eigenvalue λ of* L*, then for any $c \in \mathbb{F}$ with $c \neq 0$ the element $cX \in V$ is also an eigenvector of L associated with the eigenvalue λ. Indeed, one only has to observe that*

$$\mathrm{L}(cX) = c\mathrm{L}(X) = c\lambda X = \lambda(cX).$$

More generally one has:

Lemma 7.1.4. *The set of eigenvectors associated with the eigenvalue λ of* L *is a subspace of V.*

This vector space is called *the eigenspace* associated with the eigenvalue λ of L.

Proof. We have just seen that if X is an eigenvector of L associated with the eigenvalue λ, then cX is an eigenvector associated with the same eigenvalue. This corresponds to the second condition of the definition of a subspace of V, see Definition 3.1.5.

For the first condition of the same definition, observe that if X_1, X_2 satisfy $\mathrm{L}(X_1) = \lambda X_1$ and $\mathrm{L}(X_2) = \lambda X_2$, then one has

$$\mathrm{L}(X_1 + X_2) = \mathrm{L}(X_1) + \mathrm{L}(X_2) = \lambda X_1 + \lambda X_2 = \lambda(X_1 + X_2),$$

which corresponds to this condition. □

Example 7.1.5. *Let $\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{R})$ and consider the corresponding map $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$. Then* 0 *and* 3 *are eigenvalues of $\mathrm{L}_{\mathcal{A}}$, with E_1 an eigenvector associated with the eigenvalue* 0*, and any $cE_2 + dE_3$, with $c, d \in \mathbb{R}$, an eigenvector associated with the eigenvalue* 3*. Note that the eigenspace associated with the eigenvalue* 0 *is of dimension* 1 *while the eigenspace associated with the eigenvalue* 3 *is of dimension* 2*.*

The following result is important, especially in relation with quantum mechanics.

Theorem 7.1.6. *Let $\lambda_1, \dots, \lambda_m$ be eigenvalues of* L*, and let $X_1, \dots, X_m$ be corresponding eigenvectors. If $\lambda_i \neq \lambda_j$ for any $i \neq j$, then the vectors $X_1, \dots, X_m$ are linearly independent.*

Proof. This proof is a proof by induction. Clearly, if $m = 1$ then the only eigenvector $X_1 \neq \mathbf{0}$ is linearly independent. So, let us assume that the statement is true for a certain $m - 1 \geq 1$, and let us prove it for m. Thus, let us assume that $X_1, \dots, X_{m-1}$ are linearly independent, and show that $X_1, \dots X_m$ are also linearly independent. For this purpose, consider the linear combination

$$c_1 X_1 + c_2 X_2 + \cdots + c_m X_m = \mathbf{0}, \tag{7.1.1}$$

for some coefficients $c_j \in \mathbb{F}$. By multiplying this equality by λ_m one gets

$$c_1 \lambda_m X_1 + c_2 \lambda_m X_2 + \cdots + c_m \lambda_m X_m = \mathbf{0}. \tag{7.1.2}$$

On the other hand, by applying L to (7.1.1) one gets

$$c_1\mathrm{L}(X_1) + c_2\mathrm{L}(X_2) + \cdots + c_m\mathrm{L}(X_m) = c_1\lambda_1 X_1 + c_2\lambda_2 X_2 + \cdots + c_m\lambda_m X_m = \mathbf{0}. \quad (7.1.3)$$

Finally, by subtracting (7.1.3) to (7.1.2) one obtains

$$c_1 \underbrace{(\lambda_m - \lambda_1)}_{\neq 0} X_1 + c_2 \underbrace{(\lambda_m - \lambda_2)}_{\neq 0} X_2 + \cdots + c_{m-1} \underbrace{(\lambda_m - \lambda_{m-1})}_{\neq 0} X_{m-1} = \mathbf{0}.$$

Since $X_1, \ldots, X_{m-1}$ are linearly independent, it follows that $c_1 = c_2 = \cdots = c_{m-1} = 0$. We then conclude from (7.1.1) that $c_m = 0$ as well, meaning that $X_1, \ldots, X_m$ are linearly independent. ☐

Corollary 7.1.7. *If $\mathcal{A} \in M_n(\mathbb{F})$, then the linear map $\mathrm{L}_{\mathcal{A}} : \mathbb{F}^n \to \mathbb{F}^n$ can have at most n distinct eigenvalues.*

Proof. If $\mathrm{L}_{\mathcal{A}}$ had $m > n$ eigenvalues, then the eigenvectors $X_1, \ldots, X_m$ would be a family of m linearly independent elements of $\mathbb{F}^n$, which is impossible. ☐

7.2 The characteristic polynomial

If V is a vector space over a field $\mathbb{F}$, and if $\mathrm{L} : V \to V$ is a linear map, how can one find out the set of eigenvalues of L ? In this section, we shall answer this question.

Theorem 7.2.1. *Assume that V is a finite dimensional vector space over $\mathbb{F}$, and let $\mathrm{L} : V \to V$ be linear. Then $\lambda \in \mathbb{F}$ is an eigenvalue of L if and only if $\mathrm{L} - \lambda\mathbf{1}$ is not invertible.*

Proof. If λ is an eigenvalue of L, with $X \in V$ an associated eigenvector, then

$$[\mathrm{L} - \lambda\mathbf{1}](X) = \mathrm{L}(X) - \lambda X = \lambda X - \lambda X = \mathbf{0},$$

and therefore $X \in \mathrm{Ker}(\mathrm{L} - \lambda\mathbf{1})$. By Theorem 4.7.8, it follows that $\mathrm{L} - \lambda\mathbf{1}$ is not invertible.

Reciprocally, if $\mathrm{L} - \lambda\mathbf{1}$ is not invertible, it follows from the same theorem that there exists $X \in \mathrm{Ker}(\mathrm{L} - \lambda\mathbf{1})$ with $X \neq \mathbf{0}$. In other words, there exists $X \in V$ with $X \neq \mathbf{0}$ such that $\mathrm{L}(X) - \lambda X = \mathbf{0}$, which means that $\mathrm{L}(X) = \lambda X$. Thus, λ is an eigenvalue of L and X is an associated eigenvector. ☐

Let us consider a special case of the previous statement. If $V = \mathbb{F}^n$ and $\mathrm{L} = \mathrm{L}_{\mathcal{A}}$ for some $\mathcal{A} \in M_n(\mathbb{F})$ one infers that $\lambda \in \mathbb{F}$ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ if and only if $\mathrm{L}_{\mathcal{A}} - \lambda\mathbf{1}$ is not invertible, *i.e.* if and only if $\mathcal{A} - \lambda\mathbf{1}_n$ is not invertible. However, we have seen that this holds if and only if $\mathrm{Det}(\mathcal{A} - \lambda\mathbf{1}_n) = 0$. We have thus proved:

Corollary 7.2.2. *Let $\mathbb{F}$ be an arbitrary field, and let $\mathcal{A} \in M_n(\mathbb{F})$. Then λ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ if and only if $\mathrm{Det}(\mathcal{A} - \lambda\mathbf{1}_n) = 0$.*

Definition 7.2.3. *For any $\mathcal{A} \in M_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$, one sets*

$$P_{\mathcal{A}}(\lambda) := \mathrm{Det}(\mathcal{A} - \lambda \mathbf{1}_n)$$

and call it the characteristic polynomial *associated with $\mathcal{A}$.*

Note that some authors use the following definition: $P_{\mathcal{A}}(\lambda) = \mathrm{Det}(\lambda \mathbf{1}_n - \mathcal{A})$ which is equal to $\pm\mathrm{Det}(\mathcal{A} - \lambda \mathbf{1}_n)$, depending if n is even or odd. Note also that if $\mathcal{A} \in M_n(\mathbb{F})$, then $P_{\mathcal{A}}$ is a polynomial of degree n. As a consequence of the previous corollary, one has obtained:

Proposition 7.2.4. *For any $\mathcal{A} \in M_n(\mathbb{F})$, the scalar $\lambda \in \mathbb{F}$ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ if and only if $P_{\mathcal{A}}(\lambda) = 0$.*

Examples 7.2.5. *(i) Let $\mathcal{A} = \left(\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right)$, then*

$$P_{\mathcal{A}}(\lambda) = \mathrm{Det}(\mathcal{A} - \lambda \mathbf{1}_2) = \mathrm{Det}\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 8 = (\lambda-5)(\lambda+1).$$

Thus, the eigenvalues of $\mathrm{L}_{\mathcal{A}}$ are -1 and 5.

(ii) For $\mathcal{A} = \left(\begin{smallmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{smallmatrix}\right)$ one has

$$P_{\mathcal{A}}(\lambda) = \mathrm{Det}\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 5-\lambda & -1 \\ 0 & 0 & 7-\lambda \end{pmatrix} = (1-\lambda)(5-\lambda)(7-\lambda),$$

and the eigenvalues of $\mathrm{L}_{\mathcal{A}}$ are 1, 5 and 7.

(iii) For $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ one has $P_{\mathcal{A}}(\lambda) = (1-\lambda)(1+\lambda)$, and the eigenvalues are -1 and 1.

(iv) For $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ one has $P_{\mathcal{A}}(\lambda) = \lambda^2 + 1$ and the eigenvalues are... ?

Note that once the eigenvalues have been determined, it is possible to find the eigenvectors (or the eigenspaces) by solving a linear system. Indeed, if λ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ one looks for some $X \in \mathbb{F}^n$ such that $\mathcal{A}X = \lambda X \Leftrightarrow (\mathcal{A} - \lambda \mathbf{1}_n)X = \mathbf{0}$.

Examples 7.2.6. *(i) For $\mathcal{A} = \left(\begin{smallmatrix} 1 & 2 \\ 4 & 3 \end{smallmatrix}\right)$ and $\lambda = 5$, one has to solve*

$$\left[\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is equivalent to

$$\begin{cases} -4x + 2y = 0 \\ 4x - 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x \text{ arbitrary} \\ y = 2x \end{cases}.$$

Thus, the eigenspace associated with the eigenvalue 5 is given by $\{\left(\begin{smallmatrix} x \\ 2x \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$ or equivalently $\{x\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$.

(ii) For $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and the eigenvalue* $\lambda = 1$ *one has to solve*

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is equivalent to the single equation $-x + y = 0$*, or equivalently to* $x = y$*. Thus, the eigenspace associated with the eigenvalue* 1 *is* $\{x \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \mid x \in \mathbb{R}\}$*.*

Let us now come back to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with $P_{\mathcal{A}}(\lambda) = \lambda^2 + 1$. Assume for a while that there exists λ, solution of $\lambda^2 + 1 = 0$, or equivalently $\lambda^2 = -1$. One can then wonder about the corresponding eigenspace ? For that purpose, consider

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$\left\{ \begin{array}{l} -\lambda x + y = 0 \\ -x - \lambda y = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y + \lambda^2 y = 0 \\ x = -\lambda y \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y(1 + \lambda^2) = 0 \\ x = -\lambda y \end{array} \right. .$$

Since $1 + \lambda^2 = 0$, the element y can be chosen arbitrarily, and then one can define x by the relation $x = -\lambda y$. Thus, the eigenspace associated with the eigenvalue λ is $\{y \left(\begin{smallmatrix} -\lambda \\ 1 \end{smallmatrix}\right) \mid y \in \mathbb{R}\}$ which is a one dimensional vector space. Everything looks fine, except that there is no $\lambda \in \mathbb{R}$ satisfying $\lambda^2 + 1 = 0$! At this point, it is necessary to introduce the notion of complex numbers, which will be done in the last chapter.

As a final example, one can consider the matrix $\mathcal{A} = \left(\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{smallmatrix}\right)$ with corresponding characteristic polynomial $P_{\mathcal{A}}(\lambda) = (2 - \lambda)^2(3 - \lambda)$. Thus, the eigenvalues of $\mathrm{L}_{\mathcal{A}}$ are 2 and 3. It is good exercise to check this characteristic polynomial, and to determine the eigenspace corresponding to these eigenvalues, see Exercise 7.8.

We can now define an important set related to each linear map.

Definition 7.2.7. *Let* V *be a finite dimensional vector space, and let* $\mathrm{L} : V \to V$ *be a linear map. The set of all eigenvalues of* L *is called* the spectrum *of* L *and is denoted by* $\sigma(L)$*,* i.e. $\sigma(L) = \{\lambda_1, \lambda_2, \dots\}$ *with each* λ_j *an eigenvalue of* L*.*

Before the next statement, let us remind that if $\mathcal{B}$ is an invertible matrix, then one has

$$1 = \mathrm{Det}(\mathbf{1}_n) = \mathrm{Det}\big(\mathcal{B}\mathcal{B}^{-1}\big) = \mathrm{Det}(\mathcal{B})\mathrm{Det}\big(\mathcal{B}^{-1}\big)$$

which means that $\mathrm{Det}(\mathcal{B}^{-1}) = \mathrm{Det}(\mathcal{B})^{-1}$.

Lemma 7.2.8. *Let* $\mathcal{A} \in M_n(\mathbb{F})$ *and consider* $\mathrm{L}_{\mathcal{A}} : \mathbb{F}^n \to \mathbb{F}^n$ *the associated linear map. Let* $\mathcal{B} \in M_n(\mathbb{F})$ *be invertible. Then*

$$\sigma\big(\mathrm{L}_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}}\big) = \sigma\big(\mathrm{L}_{\mathcal{A}}\big).$$

Proof. One has

$$\begin{aligned}\mathrm{Det}\big(\mathcal{B}\mathcal{A}\mathcal{B}^{-1}-\lambda\mathbf{1}_n\big)&=\mathrm{Det}\big(\mathcal{B}\mathcal{A}\mathcal{B}^{-1}-\lambda\mathcal{B}\mathbf{1}_n\mathcal{B}^{-1}\big)=\mathrm{Det}\big(\mathcal{B}(\mathcal{A}-\lambda\mathbf{1}_n)\mathcal{B}^{-1}\big)\\&=\mathrm{Det}(\mathcal{B})\mathrm{Det}(\mathcal{A}-\lambda\mathbf{1}_n)\mathrm{Det}(\mathcal{B}^{-1})=\mathrm{Det}(\mathcal{A}-\lambda\mathbf{1}_n).\end{aligned}$$

Thus, λ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ if and only if λ is an eigenvalue of $\mathrm{L}_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}}$. □

Lemma 7.2.9. *For any $\mathcal{A}\in M_n(\mathbb{F})$, $\lambda\in\mathbb{F}$ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ if and only if λ is an eigenvalue of $\mathrm{L}_{{}^t\mathcal{A}}$.*

Proof. It is sufficient to observe that $({}^t\mathcal{A}-\lambda\mathbf{1}_n)={}^t(\mathcal{A}-\lambda\mathbf{1}_n)$ and to recall that $\mathrm{Det}(\mathcal{B})=\mathrm{Det}({}^t\mathcal{B})$ for any $\mathcal{B}\in M_n(\mathbb{F})$, see Lemma 6.2.6. □

7.3 Eigenvalues and eigenfunctions for symmetric matrices

The aim of this section is to show that if $\mathcal{A}\in M_n(\mathbb{R})$ is symmetric, *i.e.* ${}^t\mathcal{A}=\mathcal{A}$, then the corresponding linear map $\mathrm{L}_{\mathcal{A}}$ has n eigenvalues $\lambda_1,\dots,\lambda_n$ (some of them can be equal) and n mutually orthogonal eigenvectors. In fact, we shall prove a slightly more general statement, valid for more general linear maps.

First of all, recall that if ${}^t\mathcal{A}=\mathcal{A}$, then the corresponding bilinear map $\mathrm{F}_{\mathcal{A}}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and defined by $\mathrm{F}_{\mathcal{A}}(X,Y)={}^tX\mathcal{A}Y$ is symmetric. In other word, it means that $\mathrm{F}_{\mathcal{A}}(X,Y)=\mathrm{F}_{\mathcal{A}}(Y,X)$, see Exercise 5.6.

Lemma 7.3.1. *If $\mathcal{A}\in M_n(\mathbb{R})$ is symmetric, and if $\lambda_1,\lambda_2\in\mathbb{R}$ are eigenvalues of $\mathrm{L}_{\mathcal{A}}$ with $\lambda_1\neq\lambda_2$, then any associated eigenvectors X_1 and X_2 satisfy $X_1\perp X_2$.*

Proof. One has

$$\mathrm{F}_{\mathcal{A}}(X_1,X_2)={}^tX_1\mathcal{A}X_2={}^tX_1(\lambda_2X_2)=\lambda_2{}^tX_1X_2=\lambda_2(X_1\cdot X_2)$$

since $\mathcal{A}X_2=\lambda_2X_2$. Here $(X_1\cdot X_2)$ means the scalar product between the two vectors X_1 and X_2. However, since $\mathrm{F}_{\mathcal{A}}$ is symmetric one also has

$$\mathrm{F}_{\mathcal{A}}(X_1,X_2)=\mathrm{F}_{\mathcal{A}}(X_2,X_1)={}^tX_2\mathcal{A}X_1={}^tX_2(\lambda_1X_1)=\lambda_1{}^tX_2X_1=\lambda_1(X_2\cdot X_1)$$

since $\mathcal{A}X_1=\lambda_1X_1$. By comparing these expressions, one has thus obtained that

$$\lambda_2(X_1\cdot X_2)=\lambda_1(X_2\cdot X_1).$$

However, since $X_1\cdot X_2=X_2\cdot X_1$ and since $\lambda_1\neq\lambda_2$ one concludes that $X_1\cdot X_2=0$, which means that the two vectors are orthogonal. □

Let us now observe that if $\mathcal{A} \in M_n(\mathbb{R})$ and if λ is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ with the corresponding eigenspace of dimension m, then one can always choose m mutually orthogonal elements $X_1, \dots, X_m$ which satisfy $\mathrm{L}_{\mathcal{A}}(X_j) = \lambda X_j$ for $j \in \{1, \dots, m\}$. Indeed, if we denote by V_λ the eigenspace associated with the eigenvalue λ, we can apply Graham-Schmidt to this subspace and obtain a basis of V_λ containing m elements. Each of these elements still satisfies $\mathrm{L}_{\mathcal{A}}(X_j) = \lambda X_j$. Note that the dimension of the eigenspace V_λ is called *the geometric multiplicity* of the eigenvalue λ.

Theorem 7.3.2. *Let $\mathcal{A} \in M_n(\mathbb{R})$, and assume that there exists $X_1, \dots, X_n \in \mathbb{R}^n$, with $X_j \neq 0$ and such that $\mathrm{L}_{\mathcal{A}}(X_j) = \lambda_j X_j$ for some $\lambda_j \in \mathbb{R}$ and all $j \in \{1, \dots, n\}$. Assume also that $\mathrm{Vect}(X_1, \dots, X_n) = \mathbb{R}^n$. Then if one defines the matrix $\mathcal{B}$ with the column $\mathcal{B}^j$ given by $\mathcal{B}^j = X_j$, it follows that $\mathcal{B}$ is invertible and that*

$$\mathcal{B}^{-1}\mathcal{A}\mathcal{B} = \mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

where $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ corresponds to the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$ on its diagonal.

Remark 7.3.3. *(i) We shall prove subsequently that the assumptions of this theorem are satisfied whenever $\mathcal{A}$ is symmetric. The assumptions are also satisfied if $\mathcal{A}$ is arbitrary but $\mathrm{L}_{\mathcal{A}}$ has n distinct eigenvalues, see Theorem 7.1.6.*

(ii) If we consider $\mathcal{B}$ as a change of bases, then the statement means that in the basis defined by the vectors $X_1, \dots, X_n$, the linear map $\mathrm{L}_{\mathcal{B}^{-1}\mathcal{A}\mathcal{B}}$ is diagonal.

Proof. Since $X_1, \dots, X_n$ are linearly independent, it follows that $\mathrm{Det}(\mathcal{B}) \neq 0$ and thus that $\mathcal{B}$ is invertible, with inverse denoted by $\mathcal{B}^{-1}$.

Let us now compute

$$\begin{aligned}\mathcal{B}^{-1}\mathcal{A}\mathcal{B} &= \mathcal{B}^{-1}\mathcal{A}\big(X_1\ X_2\ \dots\ X_n\big) = \mathcal{B}^{-1}\big(\mathcal{A}X_1\ \mathcal{A}X_2\ \dots\ \mathcal{A}X_n\big) \\ &= \mathcal{B}^{-1}\big(\lambda_1 X_1\ \lambda_2 X_2\ \dots\ \lambda_n X_n\big) = \mathcal{B}^{-1}\mathcal{B}\ \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \mathrm{diag}(\lambda_1, \dots, \lambda_n).\end{aligned}$$

Indeed, observe that

$$\begin{aligned}\big((X_1\ X_2\ \dots\ X_n)\ \mathrm{diag}(\lambda_1, \dots, \lambda_n)\big)_{ij} &= \sum_{k=1}^n (X_1\ X_2\ \dots\ X_n)_{ik}\ \mathrm{diag}(\lambda_1, \dots, \lambda_n)_{kj} \\ &= (X_1\ X_2\ \dots\ X_n)_{ij}\ \lambda_j \\ &= (\lambda_1 X_1\ \lambda_2 X_2\ \dots\ \lambda_n X_n)_{ij}\end{aligned}$$

since $\mathrm{diag}(\lambda_1, \dots, \lambda_n)_{kj} = \lambda_j$ if $k = j$ and 0 otherwise. □

From now on, we shall establish a link between the eigenvalues/eigenvectors and a geometric construction. For that purpose and for any symmetric matrix $\mathcal{A} \in M_n(\mathbb{R})$ let us define $f_{\mathcal{A}} : \mathbb{R}^n \to \mathbb{R}$ by

$$f_{\mathcal{A}}(X) := \mathrm{F}_{\mathcal{A}}(X, X) = {}^tX\mathcal{A}X,$$

and call it *the quadratic form* associated with $\mathcal{A}$.

Examples 7.3.4. *(i) If* $\mathcal{A} = \left(\begin{smallmatrix} 3 & -1 \\ -1 & 3 \end{smallmatrix}\right)$*, then*

$$f_{\mathcal{A}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3x_1^2 - 2x_1x_2 + 3x_2^2.$$

(ii) More generally, if $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{R})$ *with* $\mathcal{A}$ *symmetric, then*

$$f_{\mathcal{A}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1\ \dots\ x_n)\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i,j=1}^{n} a_{ij}\ x_i\ x_j.$$

Let us now consider the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, *i.e.*

$$\mathbb{S}^{n-1} = \{X \in \mathbb{R}^n \mid \|X\| = 1\},$$

and for a symmetric matrix $\mathcal{A} \in M_n(\mathbb{R})$ we consider $f_{\mathcal{A}}(X)$ with $X \in \mathbb{S}^{n-1}$.

Definition 7.3.5. *A point* $X \in \mathbb{S}^{n-1}$ *is a maximum for* $f_{\mathcal{A}}$ *on* $\mathbb{S}^{n-1}$ *if* $f_{\mathcal{A}}(X) \geq f_{\mathcal{A}}(Y)$ *for any* $Y \in \mathbb{S}^{n-1}$.

Note that such a maximum always exists, but it can be non-unique. For example if $\mathcal{A} = \mathbf{1}_n$, then

$$f_{\mathcal{A}}(X) = f_{\mathbf{1}_n}(X) = {}^tX\mathbf{1}_nX = X \cdot X = \|X\|^2 = 1$$

and thus $f_{\mathbf{1}_n}$ is constant on the sphere. It means that any $X \in \mathbb{S}^{n-1}$ is a maximum for $f_{\mathbf{1}_n}$ on $\mathbb{S}^{n-1}$.

The following result establishes a link between the eigenvalues of $\mathrm{L}_{\mathcal{A}}$ and the maximum points of $f_{\mathcal{A}}$.

Theorem 7.3.6. *If* $\mathcal{A} \in M_n(\mathbb{R})$ *is symmetric and if* X *is a maximum for* $f_{\mathcal{A}}$ *on* $\mathbb{S}^{n-1}$*, then the value* $f_{\mathcal{A}}(X)$ *is an eigenvalue for* $\mathrm{L}_{\mathcal{A}}$ *with a corresponding eigenvector* X*,* i.e.

$$\mathrm{L}_{\mathcal{A}}(X) = \mathcal{A}X = f_{\mathcal{A}}(X)\,X.$$

Proof. Let $H_{\mathbf{0},X} = \{Y \in \mathbb{R}^n \mid Y \cdot X = 0\}$ be the hyperplane perpendicular to X, of dimension $n-1$, and let us choose any $Y \in H_{\mathbf{0},X}$ with $\|Y\| = 1$. For any $t \in \mathbb{R}$, one sets

$$C(t) := \cos(t)\,X + \sin(t)\,Y \quad \in \mathbb{R}^n.$$

Observe that since $X \cdot Y = 0$ one has

$$\|C(t)\|^2 = \|\cos(t)\,X\|^2 + \|\sin(t)\,Y\|^2 = \cos^2(t)\,\|X\|^2 + \sin^2(t)\,\|Y\| = \cos^2(t) + \sin^2(t) = 1.$$

It follows that for any $t \in \mathbb{R}$ the point $C(t)$ belongs to $\mathbb{S}^{n-1}$, and in addition one has $C(0) = X$. In more precise words, the map

$$\mathbb{R} \ni t \mapsto C(t) \in \mathbb{S}^{n-1}$$

is a curve on $\mathbb{S}^{n-1}$ passing through X for $t = 0$. Let us also observe that

$$C'(t) = -\sin(t)\,X + \cos(t)\,Y$$

and that $C'(0) = Y$. Note that this latter quantity corresponds to the direction of the curve at $t = 0$

Consider now the map $\mathbb{R} \ni t \mapsto f_{\mathcal{A}}\big(C(t)\big) \equiv {}^tC(t)\mathcal{A}C(t) \quad \in \mathbb{R}$. Since $f_{\mathcal{A}}(X)$ is maximal and since $C(0) = X$, this map $t \mapsto f_{\mathcal{A}}\big(C(t)\big)$ is (locally) maximal at $t = 0$, and thus $f_{\mathcal{A}}\big(C(t)\big)'\big|_{t=0} = 0$. Since one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big({}^tC(t)\mathcal{A}C(t)\big)\big|_{t=0} &= \big({}^tC'(t)AC(t) + {}^tC(t)\mathcal{A}C'(t)\big)\big|_{t=0} \\
&= {}^tY\mathcal{A}X + {}^tX\mathcal{A}Y \\
&= 2{}^tY\mathcal{A}X,
\end{aligned}$$

where we have used that ${}^tY\mathcal{A}X = {}^tX\mathcal{A}Y$ (see Exercise 5.6), it follows that ${}^tY\mathcal{A}X = 0$ for any $Y \in H_{\mathbf{0},Y}$. In addition, since ${}^tY\mathcal{A}X = Y \cdot (\mathcal{A}X)$, one infers that $\mathcal{A}X \in H_{\mathbf{0},X}^{\perp}$, and consequently that $\mathcal{A}X = \lambda X$ for some $\lambda \in \mathbb{R}$ (recall that $H_{\mathbf{0},X}$ is of dimension $n-1$ and thus that only $\mathrm{Vect}(X)$ is perpendicular to it).

Finally, one observes that since $\|X\| = 1$ one has

$$f_{\mathcal{A}}(X) = {}^tX\mathcal{A}X = X \cdot (\mathcal{A}X) = X \cdot (\lambda X) = \lambda\|X\|^2 = \lambda$$

which means that $\mathrm{L}_{\mathcal{A}}(X) = \mathcal{A}X = f_{\mathcal{A}}(X)X$, as expected. □

Let us observe that by using the notation introduced in Chapter 5 one has

$$f_{\mathcal{A}}(X) = {}^tX\mathcal{A}X = X \cdot (\mathcal{A}X) = \langle X, \mathcal{A}X\rangle = \langle X, \mathrm{L}_{\mathcal{A}}(X)\rangle$$

and that

$$H_{\mathbf{0},X} = \{Y \in \mathbb{R}^n \mid Y \cdot X = 0\} = \{Y \in \mathbb{R}^n \mid \langle Y, X\rangle = 0\}.$$

Thus, what really matters in the previous statement and its proof is the existence of a scalar product, and that $\langle Y, \mathrm{L}_{\mathcal{A}}(X)\rangle = \langle \mathrm{L}_{\mathcal{A}}(Y), X\rangle$ (which is a more general formulation of the equality ${}^tY\mathcal{A}X = {}^tX\mathcal{A}Y$). By using this observation, one can easily generalize the previous proof and statement. For that purpose, let us first provide a new definition.

Definition 7.3.7. *Let V be a vector space and let $\langle\cdot,\cdot\rangle$ be a scalar product on V. A linear map $\mathrm{L} : V \to V$ is* symmetric with respect to the scalar product *if it satisfies*

$$\langle Y, \mathrm{L}(X)\rangle = \langle \mathrm{L}(Y), X\rangle \qquad \forall X, Y \in V.$$

Theorem 7.3.8. *Let V be a finite dimensional vector space endowed with a scalar product, and let $\mathrm{L} : V \to V$ be a linear map which is symmetric with respect to the scalar product. Then L possess an eigenvalue, with eigenvector $X \neq \mathbf{0}$.*

Definition 7.3.9. *Let V be a vector space and $\mathrm{L} : V \to V$ be a linear map. A subspace $W \subset V$ is* stable for L *if* $\mathrm{L}(W) \subset W$*, i.e. if whenever $X \in W$ then $\mathrm{L}(X) \in W$.*

Examples 7.3.10. *(i) $\{\mathbf{0}\}$ and V are always stable for any linear map $\mathrm{L} : V \to V$,*

(ii) $\mathrm{Ker}(\mathrm{L})$ is stable since for any $X \in \mathrm{Ker}(\mathrm{L})$ one has $\mathrm{L}(X) = \mathbf{0} \in \mathrm{Ker}(\mathrm{L})$,

(iii) If W is the eigenspace associated with an eigenvalue λ of L, then W is stable.

For the next statement, recall that if W is a subspace of a vector space V endowed with a scalar product, then

$$W^{\perp} = \{Y \in V \mid \langle Y, X\rangle = 0 \ \forall X \in W\}.$$

Lemma 7.3.11. *Let V be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be a scalar product on V. Let $\mathrm{L} : V \to V$ be a linear map which is symmetric with respect to the scalar product. If W is stable for L, then $W^{\perp}$ is stable for L.*

Proof. Let $Y \in W^{\perp}$ and $X \in W$, then $\langle \mathrm{L}(Y), X\rangle = \langle Y, \mathrm{L}(X)\rangle = 0$ since $\mathrm{L}(X) \in W$. Thus $\mathrm{L}(Y) \in W^{\perp}$ for any $Y \in W^{\perp}$, which means precisely that $W^{\perp}$ is stable. □

We can now state and prove the most important result of this section.

Theorem 7.3.12. *Let V be a vector space of dimension n and endowed with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathrm{L} : V \to V$ be a linear map which is symmetric with respect to the scalar product. Then V possesses an orthonormal basis of eigenvectors of L. In other words there exist $Y_1, \dots, Y_n$ mutually orthogonal and with $\|Y_j\|^2 = \langle Y_j, Y_j\rangle = 1$ such that $V = \mathrm{Vect}(Y_1, \dots, Y_n)$ and such that $\mathrm{L}(Y_j) = \lambda_j Y_j$ for some λ_j.*

Proof. By Theorem 7.3.8 there exists $X_1 \neq \mathbf{0}$ such that $\mathrm{L}(X_1) = \lambda_1 X_1$ for some λ_1. If one sets $W_1 = \mathrm{Vect}(X_1)$, then W is stable for L, and the same property holds for $W_1^{\perp}$. Thus $W_1^{\perp}$ is a subspace of V of dimension $n-1$, and L is a symmetric linear map in $W_1^{\perp}$ (endowed with the scalar product inherited from V). Thus, we can again apply the previous theorem in $W_1^{\perp}$ instead of in V, and there exists $X_2 \in W_1^{\perp}$ with $X_2 \neq \mathbf{0}$, such that $\mathrm{L}(X_2) = \lambda_2 X_2$. Then, by defining $W_2 := \mathrm{Vect}(X_2)$, one obtains that $W_2^{\perp}$ (the subspace orthogonal to W_2 in W_1) is of dimension $n-2$, and is stable for L. Since L is a symmetric linear map in $W_2^{\perp}$ one can go on iteratively in the procedure, up to W_n.

Finally, by fixing $Y_j := X_j/\|X_j\|$ one gets that $Y_j \in W_j$, that $\|Y_j\| = 1$ and by construction Y_j is orthogonal to Y_k whenever $j \neq k$. One has thus obtained a basis of V which satisfies the stated properties. □

Remark 7.3.13. *In the basis $\{Y_1, \dots, Y_k\}$ the linear map L is diagonal. Whenever there exists a basis such that a linear map L is diagonal is this basis, one says that L is* diagonalizable*.*

Let us summarize our findings: One has obtained that in a vector space of finite dimension and endowed with a scalar product, any symmetric linear map is diagonalizable. Equivalently, if $\mathcal{A} \in M_n(\mathbb{R})$ is symmetric, then the linear map $\mathrm{L}_\mathcal{A}$ is diagonalizable. In particular it means that if $\mathcal{A} \in M_n(\mathbb{R})$ is symmetric, then there exists $\lambda_1, \dots \lambda_n \in \mathbb{R}$ such that

$$P_\mathcal{A}(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda). \tag{7.3.1}$$

Note that all λ_j need not be different. For example, one could have $\lambda_2 = \lambda_1$ but $\lambda_3 \neq \lambda_1$. The number of times a value λ_j appears in this decomposition is called *the algebraic multiplicity* of the eigenvalue λ_j. What the previous theorem says is that if $\mathcal{A}$ is symmetric, the algebraic multiplicity of an eigenvalue is equal to the geometric multiplicity of this eigenvalue (*i.e.* to the dimension of the corresponding eigenspace). Note that this equality holds for symmetric matrices, but it is not true in general.

7.4 Complex vector spaces

In Chapter 9, the field $\mathbb{C}$ of complex numbers is recalled. Thus, one can speak about complex vector spaces, as for example $\mathbb{C}^n$, which is of dimension n. One can also freely speak about $M_n(\mathbb{C})$, *i.e.* matrices with each entry in $\mathbb{C}$.

For any $\mathcal{A} \in M_n(\mathbb{C})$, let us consider $\mathrm{L}_\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\mathrm{L}_\mathcal{A}(X) = \mathcal{A}X$ which is obviously a linear map. Then, *the fundamental theorem of algebra* says that there exist $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$P_\mathcal{A}(\lambda) = \mathrm{Det}(\mathcal{A} - \lambda \mathbf{1}_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda).$$

Note that we have already seen such a factorization in equation (7.3.1), but it was only for symmetric matrices. Here, there is no restriction on $\mathcal{A}$, but the eigenvalues λ_j can be complex. In other words, this fundamental theorem of algebra claims that counting multiplicity there always exist n solutions to the equation $P_\mathcal{A}(\lambda) = 0$. However, be careful that this factorization does not imply that any matrix $\mathcal{A}$ is diagonalizable, even on $\mathbb{C}^n$. For example, for the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, one has $P_\mathcal{A}(\lambda) = \lambda^2$ (which means that $\lambda_1 = \lambda_2 = 0$), but this matrix can not be diagonalized in any basis.

Another natural question when dealing with $\mathbb{C}^n$ is how to endow it with a scalar product ? Let us recall that a scalar product was used for defining a norm by the relation $\|X\|^2 = \sqrt{\langle X, X\rangle}$, see Definition 5.1.5. For example, if $x \in \mathbb{R}$, it is necessary that $\langle x, x\rangle \geq 0$. Thus, let us consider two complex numbers z_1, z_2 and set

$$\langle z_1, z_2\rangle := z_1 \overline{z_2}. \tag{7.4.1}$$

Then one observes that if $z = x + iy$ with $x, y \in \mathbb{R}$ one has

$$\langle z, z\rangle = (x + iy)\overline{(x + iy)} = (x + iy)(x - iy) = x^2 + y^2 \geq 0.$$

In fact, this corresponds to the (square of the) norm of z when one identifies $\mathbb{C}$ with the plane $\mathbb{R}^2$. Similarly, if $Z = {}^t(z_1, \dots, z_n) \in \mathbb{C}^n$ and $Z' = {}^t(z'_1, \dots, z'_n) \in \mathbb{C}^n$, one sets

$$\langle Z, Z' \rangle := \sum_{j=1}^{n} z_j \, \overline{z'_j} \tag{7.4.2}$$

and observes again that $\langle Z, Z \rangle \geq 0$.

In Chapter 5, the abstract notion of a scalar product was defined for real vector space. Let us complement this definition in the case of a complex vector space (but observe that the real scalar product is a special case of the following definition).

Definition 7.4.1. A scalar product *on a complex vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for any $X, Y, Z \in V$ and $\lambda \in \mathbb{C}$ one has*

(i) $\langle X, Y \rangle = \overline{\langle Y, X \rangle}$,

(ii) $\langle X + Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle$,

(iii) $\langle \lambda X, Y \rangle = \lambda \langle X, Y \rangle = \langle X, \overline{\lambda} y \rangle$,

(iv) $\langle X, X \rangle \geq 0$ *and* $\langle X, X \rangle = 0$ *if and only if* $X = \mathbf{0}$.

It is then easily observed that the definition provided in (7.4.1) and in (7.4.2) are indeed scalar product on $\mathbb{C}$ and $\mathbb{C}^n$ respectively.

Let us now consider $\mathcal{A} = (a_{ij}) \in M_n(\mathbb{C})$ and let $Z, Z' \in \mathbb{C}^n$. Then one has

$$\begin{aligned} \langle \mathrm{L}_{\mathcal{A}}(Z), Z' \rangle &= \langle \mathcal{A}Z, Z' \rangle = \sum_{j=1}^{n} (\mathcal{A}Z)_j \, \overline{Z'_j} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} Z_k \overline{Z'_j} \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} Z_k {}^t a_{kj} \overline{Z'_j} = \sum_{k=1}^{n} Z_k \Big(\sum_{j=1}^{n} \overline{\overline{{}^t a_{kj}} Z'_j} \Big) = \langle Z, \overline{{}^t\mathcal{A}} Z' \rangle \\ &= \langle Z, \mathrm{L}_{\overline{{}^t\mathcal{A}}}(Z') \rangle. \end{aligned}$$

For simplicity, let us set $\mathcal{A}^* := \overline{{}^t\mathcal{A}}$. We have thus shown that $\langle \mathrm{L}_{\mathcal{A}}(Z), Z' \rangle = \langle Z, \mathrm{L}_{\mathcal{A}^*}(Z') \rangle$.

In the next statement, we rephrase in this more precise setting what has already been obtained in Theorem 7.3.12.

Theorem 7.4.2. *If $\mathcal{A} \in M_n(\mathbb{C})$ satisfies $\mathcal{A}^* = \mathcal{A}$, then $\mathrm{L}_{\mathcal{A}}$ is diagonalizable, with n real eigenvalues λ_j.*

For completeness, let us check that the eigenvalues of $L_{\mathcal{A}}$ are real, provided $\mathcal{A}^* = \mathcal{A}$. Thus, assume that λ_j is an eigenvalue of $\mathrm{L}_{\mathcal{A}}$ with corresponding eigenvector $X_j \neq 0$ and observe that

$$\begin{aligned} \lambda_j \|X_j\|^2 &= \langle \lambda_j X_j, X_j \rangle = \langle L_{\mathcal{A}}(X_j), X_j \rangle = \langle X_j, L_{\mathcal{A}}(X_j) \rangle \\ &= \langle X_j, \lambda_j X_j \rangle = \overline{\lambda_j} \langle X_j, X_j \rangle = \overline{\lambda_j} \|X_j\|^2. \end{aligned}$$

Since $\|X_j\| \neq 0$ it follows that $\lambda_j = \overline{\lambda_j}$, which implies that λ_j is real.

Example 7.4.3. *If* $\mathcal{A} = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right) \in M_2(\mathbb{C})$*, then* $\mathcal{A}^* = \mathcal{A}$ *and one observes that* $P_{\mathcal{A}}(\lambda) = \mathrm{Det}\left(\begin{smallmatrix} -\lambda & i \\ -i & -\lambda \end{smallmatrix}\right) = (\lambda+1)(\lambda-1)$*. Thus the eigenvalue of* $\mathcal{A}$ *are real, even so* $\mathcal{A}$ *looks rather complex !*

Remark 7.4.4. *Let us stress that Theorem 7.4.2 is at the root of quantum mechanics. Indeed, in a suitable framework it says that "the observables have real spectrum".*

7.5 Exercises

Exercise 7.1. *Let* $\mathrm{P} : V \to V$ *be a linear map on a vector space* V *and assume that* P *is a projection. Show that* P *can only have two possible eigenvalues, namely* 0 *and* 1.

Exercise 7.2. *For any* $\theta \in [0, 2\pi)$, *consider the matrix* $\mathcal{A}(\theta) := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$ *and show that the corresponding linear map* $\mathrm{L}_{\mathcal{A}(\theta)} : \mathbb{R}^2 \to \mathbb{R}^2$ *always admits the eigenvalue* 1.

Exercise 7.3. *Consider the matrix* $\mathcal{A} := \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$ *and show that* 2 *and* 4 *are eigenvalues of the associated linear map* $\mathrm{L}_{\mathcal{A}}$. *What are all corresponding eigenvectors ? Similarly, consider the matrix* $\mathcal{B} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$ *and show that* 1, 5 *and* 7 *are eigenvalues of the associated linear map. Determine the corresponding eigenspaces.*

Exercise 7.4. *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *be invertible, and assume that* $\lambda \in \mathbb{R}$ *is an eigenvalue of* $\mathrm{L}_{\mathcal{A}}$ *with* $X \in \mathbb{R}^n$ *a corresponding eigenvector.*

1. *Is* X *an eigenvector of* $\mathrm{L}_{\mathcal{A}^3}$ *? If so, what is the corresponding eigenvalue ?*
2. *Is* X *an eigenvector of the linear map associated with* $\mathcal{A} + 2\mathbf{1}_n$ *? If so, what is the corresponding eigenvalue ?*
3. *Is* X *an eigenvector of* $\mathrm{L}_{4\mathcal{A}}$ *? If so, what is the corresponding eigenvalue ?*
4. *Can* λ *be equal to* 0 *?*
5. *Is* X *an eigenvector of* $\mathrm{L}_{\mathcal{A}^{-1}}$ *? If so, what is the corresponding eigenvalue ?*
6. *What can you say about* $\mathrm{Ker}(\mathrm{L}_{\mathcal{A}} - \lambda\mathbf{1})$ *?*
7. *What can you say about* $\mathrm{Det}(\mathcal{A} - \lambda\mathbf{1}_n)$ *?*

Exercise 7.5. *For any* $\mathcal{A} \in M_2(\mathbb{R})$, *show the following equality*

$$P_{\mathcal{A}}(\lambda) = \lambda^2 - \lambda\mathrm{Tr}(\mathcal{A}) + \mathrm{Det}(\mathcal{A}).$$

Exercise 7.6. *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *and assume that* $\mathcal{A}$ *has* n *eigenvalues* $\lambda_1, \dots, \lambda_n$. *Then, show the following equalities:*

(i) $\mathrm{Det}(\mathcal{A}) = \lambda_1\lambda_2 \dots \lambda_n$ *(product of the eigenvalues)*

(ii) $\mathrm{Tr}(\mathcal{A}) = \lambda_1 + \lambda_2 + \dots + \lambda_n$ *(sum of the eigenvalues)*

Exercise 7.7. *Let* $\mathcal{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$, *and consider the associated linear map* $\mathrm{L}_{\mathcal{A}} : \mathbb{R}^3 \to \mathbb{R}^3$. *Determine the eigenvalues of* $L_{\mathcal{A}}$ *and the corresponding eigenspaces.*

Exercise 7.8. *Let* $\mathcal{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$*. Determine the eigenvalues of* $\mathrm{L}_\mathcal{A}$ *and the corresponding eigenspaces.*

Exercise 7.9. *Let* $\mathcal{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$*. Determine the eigenvalues of* $\mathrm{L}_\mathcal{A}$ *and the corresponding eigenspaces. Consider then the matrix* $\mathcal{B} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ *and compute the product* $\mathcal{B}^{-1}\mathcal{A}\mathcal{B}$*. What do you observe, and how do you understand your result ?*

Exercise 7.10. *Let* $\mathcal{A} = \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$*. Determine the eigenvalues of* $\mathrm{L}_\mathcal{A}$ *and the corresponding eigenspaces.*

Exercise 7.11. *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *and consider the linear maps* $\mathrm{L}_\mathcal{A}$ *and* $\mathrm{L}_{{}^t\mathcal{A}}$*. Show that these linear maps have the same eigenvalues.*

Exercise 7.12. *Show that if* $\mathcal{A} \in M_n(\mathbb{R})$ *is orthogonal (*i.e. ${}^t\mathcal{A} = \mathcal{A}^{-1}$*), then the (real) eigenvalues of* $\mathrm{L}_\mathcal{A}$ *can only be* 1 *or* -1*.*

Exercise 7.13. *For* $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ *consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$*. Determine the eigenvalues of* $\mathrm{L}_\mathcal{A}$ *and the corresponding eigenspaces. Find the change of bases such that in the corresponding new basis this linear map becomes diagonal.*

Exercise 7.14. *Let* $\mathcal{A} \in M_n(\mathbb{R})$ *be symmetric. Show that there exists* $\mathcal{B} \in M_n(\mathbb{R})$ *such that* $\mathcal{B}^3 = \mathcal{A}$*.*

Exercise 7.15. *For a symmetric matrix* $\mathcal{A} \in M_n(\mathbb{R})$*, one says that* $\mathcal{A}$ *is* positive definite *if* $\langle \mathcal{A}X, X\rangle > 0$ *for any* $X \in \mathbb{R}^n$ *with* $X \neq \mathbf{0}$*. In fact, this is precisely the condition which makes the bilinear map* $\mathrm{F}_\mathcal{A}$ *define a scalar product, see Exercise 5.6. If* $\mathcal{A}$ *is symmetric and positive definite, show that*

1. *All eigenvalues of* $\mathrm{L}_\mathcal{A}$ *are strictly positive,*
2. $\mathcal{A}^2$ *is symmetric and positive definite,*
3. $\mathcal{A}^{-1}$ *is symmetric and positive definite.*

Exercise 7.16. *Let* $\mathcal{A} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$*. Compute* $\mathcal{A}^n$ *for* $n = 2$*,* $n = 3$*,* $n = 25$ *and* $n = \infty$*. You are allowed to use the result of Exercise 7.9.*

Exercise 7.17. *Let* $\mathcal{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$*, and consider the associated linear map* $\mathrm{L}_\mathcal{A} : \mathcal{C}^2 \to \mathcal{C}^2$*. Determine the eigenvalues of* $\mathrm{L}_\mathcal{A}$ *and the corresponding eigenspaces. Show that these eigenspaces are orthogonal.*

Exercise 7.18. *Do there exist* $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{R})$ *such that* $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \mathbf{1}_n$ *? Justify your answer. Note that the notion of trace can be useful for this exercise.*