Chapter 1

Geometric setting

In this Chapter we recall some basic notions on points or vectors in $\mathbb{R}^n$. The norm of a vector and the scalar product between two vectors are also introduced.

1.1 The Euclidean space $\mathbb{R}^n$

We set $\mathbb{N} := \{1, 2, 3, \dots\}$ for the set of *natural numbers*, also called *positive integers*, and let $\mathbb{R}$ be the set of all real numbers.

Definition 1.1.1. *One sets*

$$\mathbb{R}^n = \big\{(a_1, a_2, \dots, a_n) \mid a_j \in \mathbb{R} \textit{ for all } j \in \{1, 2, \dots, n\}\big\}^1.$$

Alternatively, an element of $\mathbb{R}^n$*, also called* a n-tuple *or* a vector*, is a collection of* n *numbers* $(a_1, a_2, \dots, a_n)$ *with* $a_j \in \mathbb{R}$ *for any* $j \in \{1, 2, \dots, n\}$*. The number* n *is called* the dimension of $\mathbb{R}^n$*.*

In the sequel, we shall often write $A \in \mathbb{R}^n$ for the vector $A = (a_1, a_2, \dots, a_n)$. With this notation, the values $a_1, a_2, \dots, a_n$ are called *the components* or *the coordinates* of A. For example, a_1 is the first component of A, or the first coordinate of A. Be aware that $(1, 3)$ and $(3, 1)$ are two different elements of $\mathbb{R}^2$. Note that one often writes (x, y) for elements of $\mathbb{R}^2$ and (x, y, z) for elements of $\mathbb{R}^3$, see Figure 1.1. However this notation is not really convenient in higher dimensions.

The set $\mathbb{R}^n$ can be endowed with two operations, *the addition* and *the multiplication by a scalar*.

Definition 1.1.2. *For any* $A, B \in \mathbb{R}^n$ *with* $A = (a_1, a_2, \dots, a_n)$ *and* $B = (b_1, b_2, \dots, b_n)$ *and for any* $\lambda \in \mathbb{R}$ *one defines the addition of* A *and* B *by*

$$A + B := (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) \in \mathbb{R}^n$$

and the multiplication of A *by the scalar* λ *by*

$$\lambda A := (\lambda a_1, \lambda a_2, \dots, \lambda a_n) \in \mathbb{R}^n.$$

[1]The vertical line | has to be read "such that".

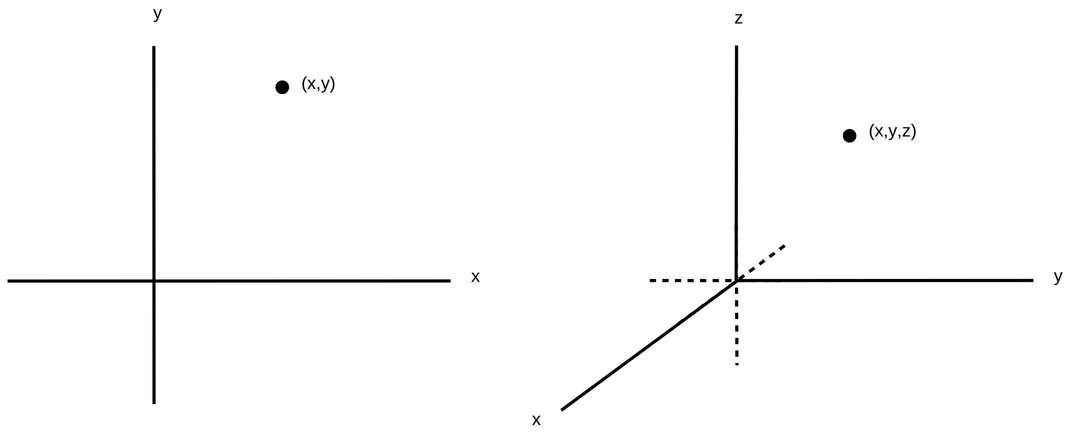

Figure 1.1: Elements of $\mathbb{R}^2$ and $\mathbb{R}^3$

Examples 1.1.3. *(i)* $(1, 3) + (2, 4) = (3, 7) \in \mathbb{R}^2$,

(ii) $(1, 2, 3, 4, 5) + (5, 4, 3, 2, 1) = (6, 6, 6, 6, 6) \in \mathbb{R}^5$,

(iii) $3(1, 2) = (3, 6) \in \mathbb{R}^2$,

(iv) $\pi(0, 0, 1) = (0, 0, \pi) \in \mathbb{R}^3$.

One usually sets

$$\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n$$

and this element satisfies $A + \mathbf{0} = \mathbf{0} + A = A$ for any $A \in \mathbb{R}^n$. If $A = (a_1, a_2, \dots, a_n)$ one also writes $-A$ for the element $-1A = (-a_1, -a_2, \dots, -a_n)$. Then, by an abuse of notation, one writes $A - B$ for $A + (-B)$ if $A, B \in \mathbb{R}^n$, and obviously one has $A - A = \mathbf{0}$. Note that $A + B$ is defined if and only if A and B belong to $\mathbb{R}^n$, but has no meaning if $A \in \mathbb{R}^n$ and $B \in \mathbb{R}^m$ with $n \neq m$.

Properties 1.1.4. *If $A, B, C \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ then one has*

(i) $A + B = B + A$, (commutativity)

(ii) $(A + B) + C = A + (B + C)$, (associativity)

(iii) $\lambda(A + B) = \lambda A + \lambda B$, (distributivity)

(iv) $(\lambda + \mu)A = \lambda A + \mu A$,

(v) $(\lambda\mu)A = \lambda(\mu A)$.

These properties will be proved in Exercise 1.3.

1.2 Located vectors in $\mathbb{R}^n$

A geometric picture can often aid our intuition (but can also be misleading). For example, one often identifies $\mathbb{R}$ with a line, $\mathbb{R}^2$ with a plane and $\mathbb{R}^3$ with the usual 3 dimensional space. In this setting, an element $A \in \mathbb{R}^n$ is often called *a point* in $\mathbb{R}^n$. However, one can also think about the elements of $\mathbb{R}^n$ as *arrows*. In this setting, the element $(3, 5) \in \mathbb{R}^2$ can be thought as an arrow starting at the point $(0, 0)$ of the usual plane with two axes and ending at the point $(3, 5)$ of this plane, see Figure 1.2. With

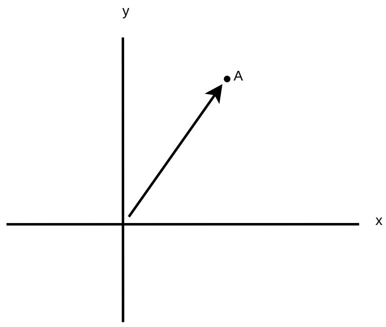

Figure 1.2: A point seen as an arrow

this interpretation in mind, the addition of two elements of $\mathbb{R}^n$ corresponds the addition of two arrows, and the multiplication by a scalar corresponds to the rescaling of an arrow, see Figure 1.3. Note that in the sequel both interpretations (points and arrows) will appear, but this should not lead to any confusion.

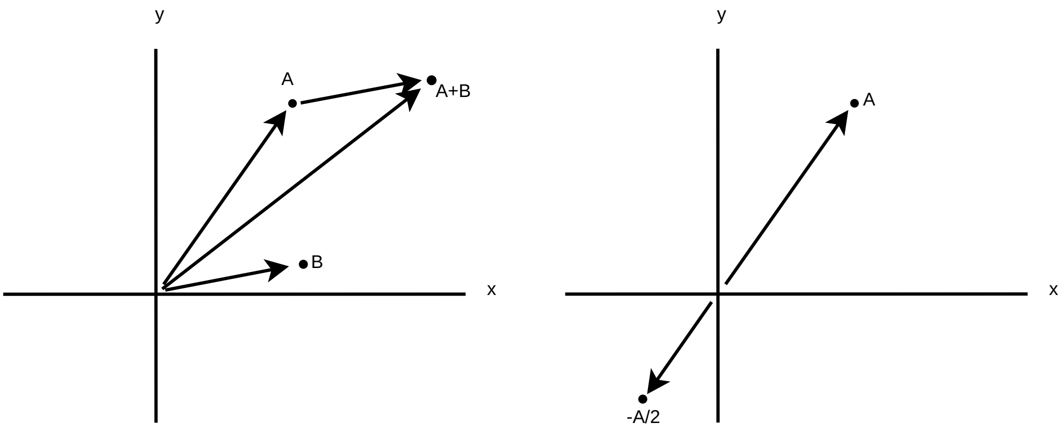

Figure 1.3: Addition of arrows and multiplication by $\lambda = -1/2$

In relation with this geometric interpretation, it is sometimes convenient to have the following notion at hand.

Definition 1.2.1. *For any $A, B \in \mathbb{R}^n$ we set $\overrightarrow{AB}$ for the arrow starting at A and ending at B, and call it the* located vector *$\overrightarrow{AB}$.*

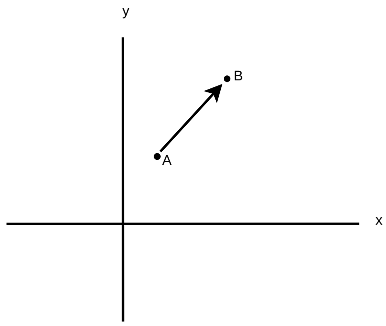

Figure 1.4: The located vector $\overrightarrow{AB}$

With this definition and for any $A \in \mathbb{R}^n$ the located vector $\overrightarrow{\mathbf{0}A}$ corresponds to the arrow mentioned in the previous geometric interpretation. For that reason, the located vector $\overrightarrow{\mathbf{0}A}$ is simply called *a vector* and is often identified with the element A of $\mathbb{R}^n$. Let us now introduce various relations between located vectors:

Definition 1.2.2. *For $A, B, C, D \in \mathbb{R}^n$, the located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are* equivalent *if $B - A = D - C$. These located vectors are* parallel *if there exists $\lambda \in \mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$ such that $B - A = \lambda(D - C)$. In particular, they have the* same direction *if $\lambda > 0$ or have* opposite direction *if $\lambda < 0$.*

In Figure 1.5 equivalent located vectors and parallel located vectors are represented. Note that the located vector $\overrightarrow{AB}$ is always equivalent to the located vector $\overrightarrow{\mathbf{0}(B - A)}$

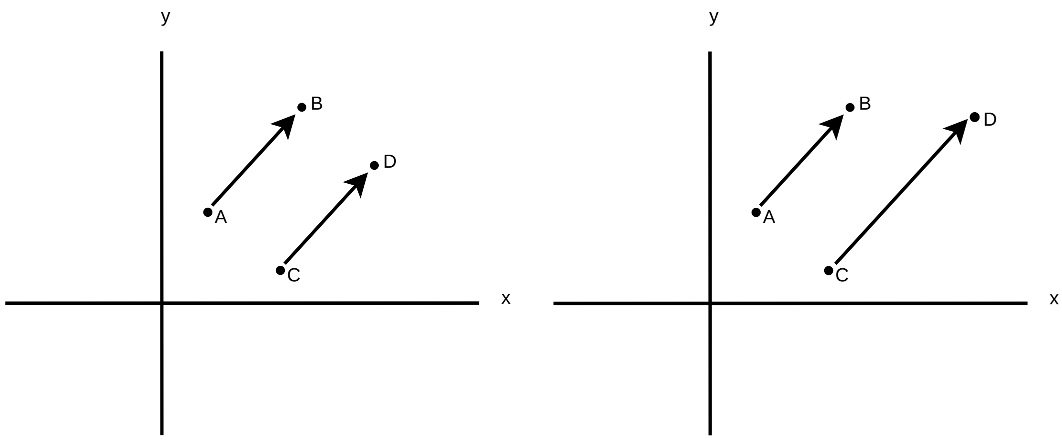

Figure 1.5: Equivalent and parallel located vectors

which is located at *the origin* $\mathbf{0}$, see Figure 1.6. This fact follows from the equality

$$(B - A) - \mathbf{0} = (B - A) = B - A.$$

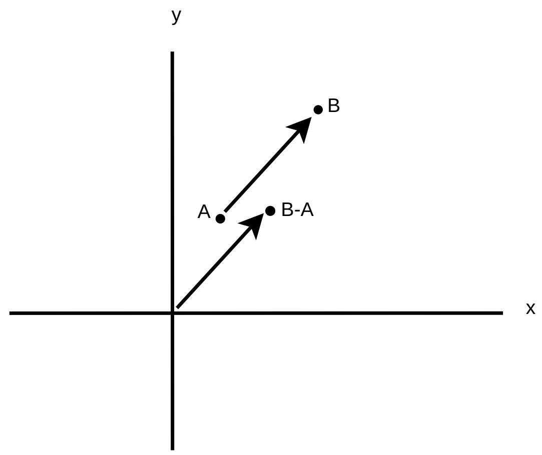

Figure 1.6: Located vector $\overrightarrow{AB}$ equivalent to the located vector $\overrightarrow{\mathbf{0}(B-A)}$

Question: What could be the meaning for two located vectors to be *perpendicular*? Even if one has an intuition in $\mathbb{R}^2$ or $\mathbb{R}^3$, one needs a precise definition for located vectors in $\mathbb{R}^n$.

1.3 Scalar product in $\mathbb{R}^n$

Definition 1.3.1. *For any $A, B \in \mathbb{R}^n$ with $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ one sets*

$$A \cdot B := a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{j=1}^{n} a_j b_j$$

and calls this number the scalar product *between A and B.*

For example, if $A = (1, 2)$ and $B = (3, 4)$, then $A \cdot B = 1{\cdot}3 + 2{\cdot}4 = 3 + 8 = 11$, but if $A = (1, 3)$ and $B = (6, -2)$, then $A \cdot B = 6 - 6 = 0$. Be aware that the previous notation is slightly misleading since the dot $\cdot$ between A and B corresponds to the scalar product while the dot between numbers just corresponds to the usual multiplication of numbers.

Properties 1.3.2. *For any $A, B, C \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ one has*

(i) $A \cdot B = B \cdot A$,

(ii) $A \cdot (B + C) = A \cdot B + A \cdot C$,

(iii) $(\lambda A) \cdot B = A \cdot (\lambda B) = \lambda(A \cdot B)$,

(iv) $A \cdot A \geq 0$, *and* $A \cdot A = 0$ *if and only if* $A = \mathbf{0}$.

These properties will be proved in Exercise 1.6.

Definition 1.3.3. *Two vectors $A, B \in \mathbb{R}^n$ are* perpendicular *or* orthogonal *if $A \cdot B = 0$, in which case one writes $A \perp B$. If $A, B, C, D \in \mathbb{R}^n$, the located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are* perpendicular *or* orthogonal *if they are equivalent to two perpendicular vectors, in which case one writes $\overrightarrow{AB} \perp \overrightarrow{CD}$.*

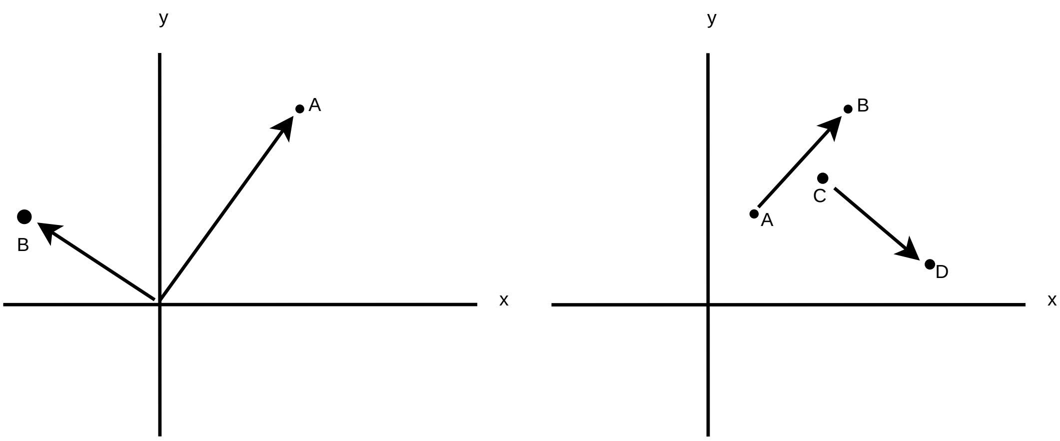

Figure 1.7: Perpendicular vectors and perpendicular located vectors

Remark first that if $A, B \in \mathbb{R}^n$ are perpendicular, then A is also perpendicular to λB for any $\lambda \in \mathbb{R}$. Indeed, from the above properties, it follows that if $A \cdot B = 0$ then $A \cdot (\lambda B) = \lambda (A \cdot B) = 0$. Now, observe also that in the setting of the previous definition, and since $\overrightarrow{AB}$ is equivalent to the vector $\overrightarrow{\mathbf{0}(B-A)}$ and since $\overrightarrow{CD}$ is equivalent to the vector $\overrightarrow{\mathbf{0}(D-C)}$, one has $\overrightarrow{AB} \perp \overrightarrow{CD}$ if and only $\overrightarrow{\mathbf{0}(B-A)}$ is perpendicular to $\overrightarrow{\mathbf{0}(D-C)}$, *i.e.* if and only if

$$(B - A) \cdot (D - C) = 0. \tag{1.3.1}$$

Example 1.3.4. *In $\mathbb{R}^n$ let us set $E_1 = (1, 0, \dots, 0)$, $E_2 = (0, 1, 0, \dots, 0)$, $\dots$, $E_n = (0, \dots, 0, 1)$ the n different vectors obtained by assigning a 1 at the coordinate j of E_j and 0 for all its other coordinates. Then, one easily checks that*

$$E_j \cdot E_k = 0 \textit{ whenever } j \neq k \quad \textit{and} \quad E_j \cdot E_j = 1 \textit{ for any } j \in \{1, 2, \dots, n\}.$$

These n vectors are said to be mutually orthogonal.

1.4 Euclidean norm in $\mathbb{R}^n$

Recall that for any $A \in \mathbb{R}^n$ one has $A^2 := A \cdot A \geq 0$.

Definition 1.4.1. *The* Euclidean norm *or simply* norm *of a vector $A \in \mathbb{R}^n$ is defined by $\|A\| := \sqrt{A^2}$. The positive number $\|A\|$ is also referred to as the* magnitude *of A. A vector of norm 1 is called a* unit vector.

Example 1.4.2. *If* $A = (-1, 2, 3) \in \mathbb{R}^3$, *then* $A \cdot A = (-1)^2 + 2^2 + 3^2 = 14$ *and therefore* $\|A\| = \sqrt{14}$.

Remark 1.4.3. *If* $n = 2$ *and in the geometric interpretation mentioned in Section 1.2, one observes that the norm* $\|A\|$ *of an element* $A \in \mathbb{R}^2$ *is compatible with Pythagoras theorem.*

Properties 1.4.4. *For any* $A \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$ *one has*

(i) $\|A\| = 0$ *if and only if* $A = \mathbf{0}$,

(ii) $\|\lambda A\| = |\lambda| \, \|A\|$,

(iii) $\| - A\| = \|A\|$.

Note that the third point is a special case of the second point. The proof of these properties will be provided in Exercise 1.8.

Definition 1.4.5. *For any* $A, B \in \mathbb{R}^n$, *the* distance *between* A *and* B, *denoted by* $d(A, B)$, *is defined by* $d(A, B) := \|B - A\|$.

Properties 1.4.6. *For any* $A, B, C \in \mathbb{R}^n$ *one has*

(i) $d(A, B) = d(B, A)$,

(ii) $d(A, B) = 0$ *if and only if* $A = B$,

(iii) $d(A - C, B - C) = d(A, B)$, *and in particular* $d(A, B) = d(\mathbf{0}, B - A)$.

The proofs of these properties are left as a free exercise. Now, keeping in mind the geometric interpretation provided in Section 1.2, it is natural to set

$$\|\overrightarrow{AB}\| := d(A, B)$$

and to call this number *the length* of the located vector $\overrightarrow{AB}$. Indeed, it follows from this definition and from Property 1.4.6.(iii) that

$$\|\overrightarrow{AB}\| = d(A, B) = d(\mathbf{0}, B - A) = \|\overrightarrow{\mathbf{0}(B - A)}\| = \|B - A\|.$$

Thus, the length of the located vector $\overrightarrow{AB}$ corresponds to the norm of the vector $(B - A) \in \mathbb{R}^n$. One also observes that any two located vectors which are equivalent have the same length.

Question: If $r > 0$ and $A \in \mathbb{R}^n$, what is

$$\{B \in \mathbb{R}^n \mid d(A,B) < r\} \quad ?$$

Can one draw a picture of this set for $n = 1$, $n = 2$ or $n = 3$?

Definition 1.4.7. *For $r > 0$ and $A \in \mathbb{R}^n$, one defines*

$$\mathscr{B}(A,r) := \{B \in \mathbb{R}^n \mid d(A,B) < r\}$$

and call $\mathscr{B}(A,r)$ the (open) ball centered at A and of radius r.

For example, if $n = 2$ then $\mathscr{B}(\mathbf{0},1)$ corresponds to the (open) unit disc in the plane, *i.e.* to the set of points on $\mathbb{R}^2$ which are at a distance strictly less than 1 from the origin $(0,0)$. If $n = 3$ then $\mathscr{B}(\mathbf{0},1)$ corresponds to the (open) unit ball in the usual 3 dimensional space.

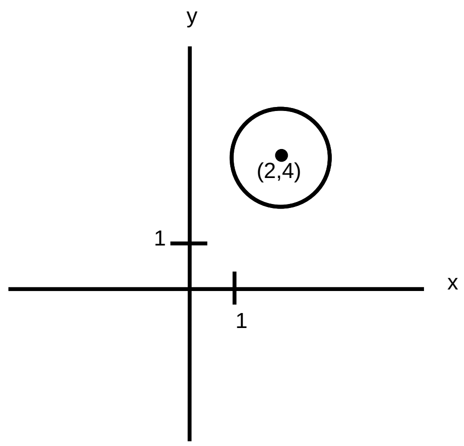

Figure 1.8: The open ball $\mathscr{B}\big((2,4),1\big)$ in $\mathbb{R}^2$

Let us now get a better intuition for the notion of orthogonal vectors. First of all, consider the following property:

Lemma 1.4.8. *For any $A, B \in \mathbb{R}^n$ one has*

$$\|B + A\| = \|B - A\| \Leftrightarrow A \cdot B = 0.$$

Proof. One has

$$\begin{aligned}
\|B + A\| = \|B - A\| &\Leftrightarrow \|B + A\|^2 = \|B - A\|^2 \\
&\Leftrightarrow (B + A) \cdot (B + A) = (B - A) \cdot (B - A) \\
&\Leftrightarrow B^2 + 2A \cdot B + A^2 = B^2 - 2A \cdot B + A^2 \\
&\Leftrightarrow 4A \cdot B = 0,
\end{aligned}$$

which justifies the statement. $\square$

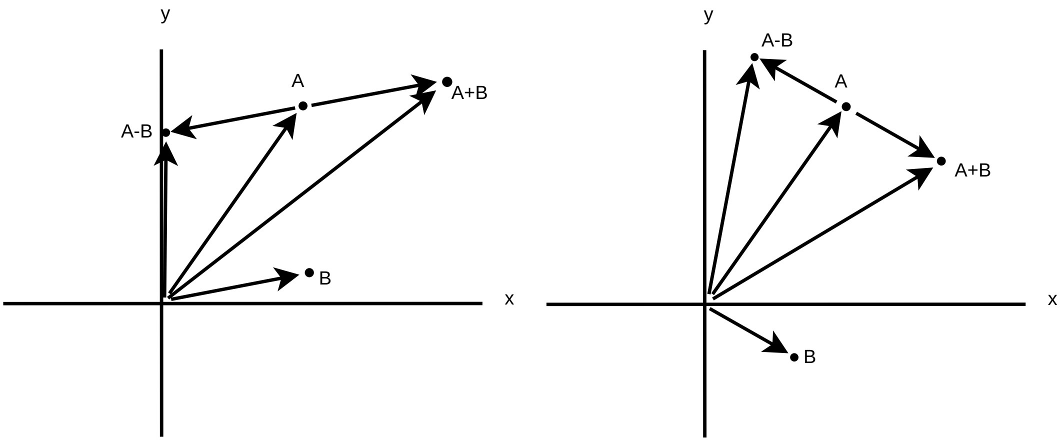

Figure 1.9: The vectors $A + B$ and $A - B$

By considering the geometric setting introduced in Section 1.2, one observes that the condition $\|B + A\| = \|B - A\|$ corresponds to our intuition for the two vectors A and B being perpendicular, see Figure 1.9. More generally, one can prove the general Pythagoras theorem:

Theorem 1.4.9. *Two vectors $A, B \in \mathbb{R}^n$ are mutually orthogonal if and only if the equality $\|A + B\|^2 = \|A\|^2 + \|B\|^2$ holds.*

The proof of this Theorem is provided in Exercise 1.9.

Question: Let $A, B \in \mathbb{R}^n$ with $B \neq \mathbf{0}$. Let P denote the point on the line passing through $\mathbf{0}$ and B, and such that the located vector $\overrightarrow{PA}$ is perpendicular to the located vector $\overrightarrow{\mathbf{0}B}$, see Figure 1.10. Clearly, $P = cB$ for some $c \in \mathbb{R}$, but how can one compute c ?

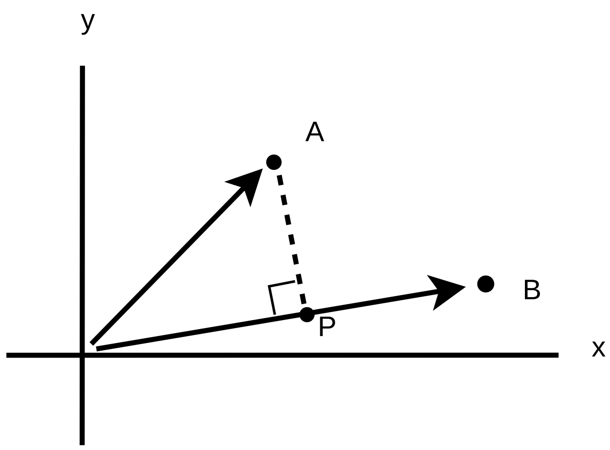

Figure 1.10:

For the answer, it is sufficient to consider the following equivalences:

$$\begin{aligned}
\overrightarrow{PA} \perp \overrightarrow{\mathbf{0}B} &\Leftrightarrow \overrightarrow{\mathbf{0}(A-P)} \perp \overrightarrow{\mathbf{0}B} \\
&\Leftrightarrow (A-P)\cdot B = 0 \\
&\Leftrightarrow (A-cB)\cdot B = 0 \\
&\Leftrightarrow A\cdot B = cB^2 \\
&\Leftrightarrow c = \frac{A\cdot B}{\|B\|^2}.
\end{aligned}$$

Definition 1.4.10. *Let $A, B \in \mathbb{R}^n$ with $B \neq \mathbf{0}$. Then* the component of A along B *is by definition the number $c := \frac{A\cdot B}{\|B\|^2}$. In this case cB is called* the orthogonal projection of A on B*.*

Let us recall from plane geometry that if one considers the right (or right-angled) triangle with vertices the points $\mathbf{0}$, A and cB with $A \neq \mathbf{0}$, $B \neq \mathbf{0}$ and with $c > 0$, then the angle θ at the vertex $\mathbf{0}$ satisfies

$$\cos(\theta) = \frac{\|cB\|}{\|A\|} = \frac{c\|B\|}{\|A\|} = \frac{(A\cdot B)\|B\|}{\|B\|^2\,\|A\|} = \frac{A\cdot B}{\|A\|\,\|B\|}.$$

Note that the same argument also holds for $c < 0$, and thus one has for any such triangle

$$\cos(\theta) = \frac{A\cdot B}{\|A\|\,\|B\|}.$$

From the above considerations and since $|\cos(\theta)| \leq 1$, one infers the following result:

Lemma 1.4.11. *For any $A, B \in \mathbb{R}^n$ one has*

$$|A\cdot B| \leq \|A\|\,\|B\|. \tag{1.4.1}$$

Let us also deduce a very useful inequality called *triangle inequality*:

Lemma 1.4.12. *For any $A, B \in \mathbb{R}^n$ one has*

$$\|A+B\| \leq \|A\| + \|B\|.$$

Proof. By taking into account the inequality (1.4.1) one obtains that

$$\begin{aligned}
\|A+B\|^2 &= A^2 + B^2 + 2A\cdot B \\
&\leq A^2 + B^2 + 2|A\cdot B| \\
&\leq A^2 + B^2 + 2\|A\|\,\|B\| \\
&= \big(\|A\| + \|B\|\big)^2.
\end{aligned}$$

The expected result is then obtained by taking the square root on both sides of the inequality. □

1.5 Parametric representation of a line

Let us consider $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$, and let $t \in \mathbb{R}$.

Question: What does $\{P + tN \mid t \in \mathbb{R}\}$ represent? Can one draw a picture of this set?

Definition 1.5.1. *For any $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$ one defines*

$$L_{P,N} := \big\{P + tN \mid t \in \mathbb{R}\big\}$$

and call this set the line passing through P and having the direction N. *More precisely $L_{P,N}$ is called* the parametric representation *of this line, see Figure 1.11.*

Remark 1.5.2. *(i) If N is replaced by λN for any $\lambda \in \mathbb{R}^*$, then $L_{P,\lambda N}$ describes the same line. In addition, any element of $L_{P,N}$ can be used instead of P and the resulting line will be the same.*

(ii) If $P, Q \in \mathbb{R}^n$, then the line passing through the two points P and Q is given by $L_{P,Q-P}$. Indeed one checks that $L_{P,Q-P} = \{P + t(Q-P) \mid t \in \mathbb{R}\}$, and that this line passes through P at $t = 0$ and passes through Q at $t = 1$.

(iii) For $P, Q \in \mathbb{R}^n$, the set $\{P + t(Q-P) \mid t \in [0,1]\}$ describes the line segment starting at P and ending at Q.

Remark 1.5.3. *If $n = 2$ a line is often describes by $\{(x,y) \in \mathbb{R}^2 \mid ax + by = c\}$ for some $a, b, c \in \mathbb{R}$. Thus, in dimension 2 a line can be described by this formulation or with $L_{P,N}$ for some $P, N \in \mathbb{R}^2$. Clearly, some relations between a, b, c and P, N can be established. However, note that the above simple description does not exist for $n > 2$ while the definition $L_{P,N}$ holds in arbitrary dimension.*

1.6 Planes and hyperplanes

Let us first recall that two located vectors are orthogonal if they are equivalent to two perpendicular vectors.

Question: Let $P, N \in \mathbb{R}^3$ with $N \neq \mathbf{0}$. How can one describe the plane passing through P and perpendicular to the direction defined by the vector N ?

For the answer, consider a point X belonging to this plane. By definition, the located vector $\overrightarrow{PX}$ is orthogonal to the located vector $\overrightarrow{\mathbf{0}N}$, or equivalently the located vector $\overrightarrow{\mathbf{0}(X-P)}$ is orthogonal to the located vector $\overrightarrow{\mathbf{0}N}$. Now this condition reads $(X-P) \perp N$, which is equivalent to $(X-P) \cdot N = 0$, or by a simple computation to the condition $X \cdot N = P \cdot N$. In summary, the plane passing through P and perpendicular to the direction defined by the vector N is given by

$$\big\{X \in \mathbb{R}^3 \mid X \cdot N = P \cdot N\big\}.$$

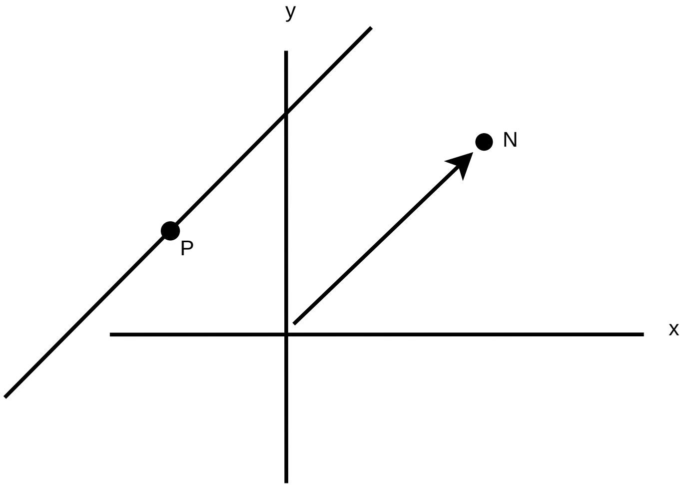

Figure 1.11: Parametric representation of a line

In this case, one also says that the plan is *normal* to the vector N.

Example 1.6.1. *If $P = (2, 1, -1)$, $N = (-1, 1, 3)$ and $X = (x, y, z)$, then*

$$X \cdot N = P \cdot N \Leftrightarrow (x, y, z) \cdot (-1, 1, 3) = -2 + 1 - 3 \Leftrightarrow -x + y + 3z = -4.$$

Therefore, the plane passing through $(2, 1, -1)$ and normal to the vector $(-1, 1, 3)$ is given by

$$\big\{(x, y, z) \in \mathbb{R}^3 \mid -x + y + 3z = -4\big\}.$$

Let us now work in arbitrary dimension.

Definition 1.6.2. *For any $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$, the set*

$$H_{P,N} := \big\{X \in \mathbb{R}^n \mid X \cdot N = P \cdot N\big\}$$

is called the hyperplane *passing through P and normal to N.*

Note that if $P = (p_1, p_2, \dots, p_n)$ and if $N = (n_1, n_2, \dots, n_n)$, then

$$H_{P,N} = \Big\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid n_1x_1 + n_2x_2 + \cdots + n_nx_n = \sum_{j=1}^{n} p_jn_j\Big\}.$$

Remark 1.6.3. *In the special case $P \cdot N = 0$, one observes that the element $\mathbf{0}$ belongs to $H_{P,N}$. Later on, we shall see that in this case $H_{P,N}$ is a* vector space*, see Chapter 3*

Properties 1.6.4. *For any $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$, and for any $\lambda \in \mathbb{R}^*$ one has*

(i) $H_{P,N} = H_{P,\lambda N}$,

(ii) If $P' \in H_{P,N}$, then $H_{P',N} = H_{P,N}$.

The proof of these properties will be provided in Exercise 1.16. It is now natural to define various notions related to hyperplanes. The following definitions correspond to the intuition we can have in $\mathbb{R}^2$ or in $\mathbb{R}^3$.

Definition 1.6.5. *Let $P, P', N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$ and with $P' \notin H_{P,N}$. Then the two hyperplanes $H_{P,N}$ and $H_{P',N}$ are* parallel.

Lemma 1.6.6. *Two parallel hyperplanes have an empty intersection.*

Proof. Let $H_{P,N}$ and $H_{P',N}$ be two parallel hyperplanes, and let us assume that there exists $X \in \mathbb{R}^n$ which belongs to both hyperplanes. This assumption means that $X \in H_{P,N}$ and $X \in H_{P',N}$, or equivalently $X\cdot N = P\cdot N$ and $X\cdot N = P'\cdot N$. As a consequence, it follows from these equalities that $P \cdot N = P' \cdot N$.

On the other hand, since the two planes are parallel, the assumption on P' is $P' \notin H_{P,N}$, which means that $P' \cdot N \neq P \cdot N$. Thus one has obtained a contradiction since $P \cdot N = P' \cdot N$ together with $P' \cdot N \neq P \cdot N$ is impossible. As a conclusion, there does not exist any X in the intersection of the two hyperplanes, or equivalently this intersection is empty. $\square$

Example 1.6.7. *For $n = 2$, $P = (0, 0)$, $P' = (0, 1)$ and $N = (1, 1)$, one checks that $P' \cdot N = 1 \neq 0 = P \cdot N$, and thus $P' \notin H_{P,N}$. In addition, if $X = (x, y)$ one easily observes that $X \in H_{P,N}$ if and only $y = -x$ while $X \in H_{P',N}$ if and only if $y = -x + 1$.*

Definition 1.6.8. *Let $P, P', N, N' \in \mathbb{R}^n$ with $N \neq \mathbf{0}$ and $N' \neq \mathbf{0}$. One defines* the angle θ between the hyperplanes $H_{P,N}$ and $H_{P',N'}$ *as the angle between their normal vectors, or more precisely*

$$\cos(\theta) := \frac{N \cdot N'}{\|N\|\,\|N'\|}.$$

From this definition, one observes that the angle between two parallel hyperplanes is equal to 0.

Observation 1.6.9. *Let $P, N \in \mathbb{R}^n$ with $N \neq \mathbf{0}$.*

(i) Since $H_{P,N} = H_{P,\lambda N}$ for any $\lambda \in \mathbb{R}^*$, one has $H_{P,N} = H_{P,\hat{N}}$ with $\hat{N} := \frac{N}{\|N\|}$. Note that $\hat{N}$ is a unit vector (see Definition 1.4.1).

(ii) The hyperplane $H_{P,N}$ divides $\mathbb{R}^n$ into two distinct regions. Indeed, for any $X \in \mathbb{R}^n$ one has either $X \cdot N > P \cdot N$, or $X \cdot N = P \cdot N$ or $X \cdot N < P \cdot N$. In the second case, X belongs to $H_{P,N}$ by definition of this hyperplane. Thus, one is left with the other two regions $\{X \in \mathbb{R}^n \mid X \cdot N > P \cdot N\}$ or $\{X \in \mathbb{R}^n \mid X \cdot N < P \cdot N\}$ and these two regions have an empty intersection.

Question: What is the distance between a point X and a hyperplane $H_{P,N}$?

The natural definition for such a notion can be understood as follows: Consider any point $Y \in H_{P,N}$ and recall that the distance $d(X,Y)$ between X and Y has been defined in Definition 1.4.5. Then, the distance $d(X, H_{P,N})$ between X and the hyperplane $H_{P,N}$ should be the minimal distance between X and any point $Y \in H_{P,N}$, namely

$$d(X, H_{P,N}) := \inf_{Y \in H_{P,N}} d(X,Y),$$

where the notation *inf* has to be read "infimum". In the next Lemma, we give an explicit formula for this distance.

Lemma 1.6.10. *For any $P, N, X \in \mathbb{R}^n$ with $N \neq \mathbf{0}$ one has*

$$d(X, H_{P,N}) = \frac{|(X-P)\cdot N|}{\|N\|} \ .$$

Proof. First of all, observe that if $X \notin H_{P,N}$, there exists $\lambda \in \mathbb{R}^*$ such that $X \cdot N - \lambda = P \cdot N$. In fact, one simply has $\lambda = X \cdot N - P \cdot N = (X-P)\cdot N$. In addition, observe that

$$X \cdot N - \lambda = P \cdot N \Leftrightarrow X \cdot N - \lambda \frac{N \cdot N}{\|N\|^2} = P \cdot N \Leftrightarrow \Big(X - \frac{\lambda}{\|N\|} \frac{N}{\|N\|} \Big) \cdot N = P \cdot N$$

which means that $X - \frac{\lambda}{\|N\|}\frac{N}{\|N\|}$ belongs to $H_{P,N}$ if $\lambda = (X-P)\cdot N$.

From this observation, one infers that $X' := X - \frac{(X-P)\cdot N}{\|N\|}\frac{N}{\|N\|} \in H_{P,N}$ and that

$$d(X,X') = \|X' - X\| = \Big\| - \frac{(X-P)\cdot N}{\|N\|} \frac{N}{\|N\|} \Big\| = \frac{|(X-P)\cdot N|}{\|N\|}.$$

As a consequence, one has $d(X, H_{P,N}) \leq \frac{|(X-P)\cdot N|}{\|N\|}$.

In order to show that this distance is the shortest one, consider any $Y \in H_{P,N}$ and use the general Pythagoras theorem for the right triangle of vertices X, Y and X'. Indeed, since $\overrightarrow{X'Y} \perp \overrightarrow{X'X}$ (because $Y \in H_{X',N}$ and $X - X' = \frac{(X-P)\cdot N}{\|N\|^2} N$) one gets:

$$d(X,Y)^2 = \|Y - X\|^2 = \|Y - X'\|^2 + \|X' - X\|^2 \geq \|X' - X\|^2 = d(X,X')^2$$

from which one infers that $d(X,Y) \geq d(X,X')$. □

Question: What are the intersections of hyperplanes? More precisely, can we find $X \in \mathbb{R}^n$ such that

$$X \in H_{P_1,N_1} \cap H_{P_2,N_2} \cap \cdots \cap H_{P_m,N_m} \ ? \tag{1.6.1}$$

Obviously, if some hyperplanes are parallel, there does not exist any X satisfying this condition. Even if the hyperplanes are not parallel, is it possible that the intersection is empty? Before answering these questions, recall once more that

$$X \in H_{P,N} \Leftrightarrow n_1 x_1 + n_2 x_2 + \cdots + n_n x_n = \sum_{j=1}^n p_j n_j.$$

and therefore equation (1.6.1) corresponds to a system of linear equations, as we shall see in the sequel.

1.7 Exercises

Exercise 1.1. *Compute $A+B$, $A-B$, $3A$ and $-2B$ in each of the following cases, and illustrate your result with the geometric interpretation.*

1. $A=(2,-1),\ B=(-1,1)$
2. $A=(2,-1,5),\ B=(-1,1,1)$
3. $A=(\pi,3,-1),\ B=(2\pi,-3,7)$

Exercise 1.2. *Let $A=(1,2)$ and $B=(3,1)$. Compute $A+2B$, $A-3B$ and $A+\frac{1}{2}B$ and provide the geometric interpretation.*

Exercise 1.3. *Write the proofs for Properties 1.1.4.*

Exercise 1.4. *In the following cases, determine which located vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are equivalent.*

1. $P=(1,-1),\ Q=(4,3),\ A=(-1,5),\ B=(5,2)$
2. $P=(1,4),\ Q=(-3,5),\ A=(5,7),\ B=(1,8)$
3. $P=(1,-1,5),\ Q=(-2,3,-4),\ A=(3,1,1),\ B=(0,5,10)$
4. $P=(2,3,-4),\ Q=(-1,3,5),\ A=(-2,3,-1),\ B=(-5,3,8)$

Similarly, determine if the located vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

1. $P=(1,-1),\ Q=(4,3),\ A=(-1,5),\ B=(7,1)$
2. $P=(1,4),\ Q=(-3,5),\ A=(5,7),\ B=(9,6)$
3. $P=(1,-1,5),\ Q=(-2,3,-4),\ A=(3,1,1),\ B=(-3,9,-17)$
4. $P=(2,3,-4),\ Q=(-1,3,5),\ A=(-2,3,-1),\ B=(-11,3,-28)$

Exercise 1.5. *Compute $A\cdot A$ and $A\cdot B$ for the following vectors.*

1. $A=(2,-1),\ B=(-1,1)$
2. $A=(2,-1,5),\ B=(-1,1,1)$
3. $A=(\pi,3,-1),\ B=(2\pi,-3,7)$
4. $A=(1,-1,1),\ B=(2,3,1)$

Which pairs of vectors are perpendicular?

Exercise 1.6. *Write the proofs for Properties 1.3.2.*

Exercise 1.7. *By using the properties of the previous exercise, show the following equalities (we use the notation A^2 for $A \cdot A$).*

1. $(A+B)^2 = A^2 + 2A \cdot B + B^2$
2. $(A-B)^2 = A^2 - 2A \cdot B + B^2$

Exercise 1.8. *Write the proofs for Properties 1.4.4.*

Exercise 1.9. *Write a proof for Theorem 1.4.9.*

Exercise 1.10. *Let us consider the pair (A, B) of elements of $\mathbb{R}^n$.*

1. $A = (2, -1), \ \ B = (-1, 1)$
2. $A = (-1, 3), \ \ B = (0, 4)$
3. $A = (2, -1, 5), \ \ B = (-1, 1, 1)$

For each pair, compute the norm of A, the norm of B, and the orthogonal projection of A along B.

Exercise 1.11. *Find the cosine between the following vectors A and B :*

1. $A = (1, 2), \ \ B = (5, 3)$
2. $A = (1, -2, 3), \ \ B = (-3, 1, 5)$

Exercise 1.12. *Determine the cosine of the angles of the triangle whose vertices are $A = (2, -1, 1)$, $B = (1, -3, -5)$ and $C = (3, -4, -4)$.*

Exercise 1.13. *Let $A_1, \dots, A_r$ be non-zero vectors of $\mathbb{R}^n$ which are all mutually perpendicular, or in other words $A_j \cdot A_k = 0$ if $j \neq k$. Let $c_1, \dots, c_r$ be real numbers such that*

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r = \mathbf{0}.$$

Show that $c_j = 0$ for all $j \in \{1, 2, \dots, r\}$.

Exercise 1.14. *Find a parametric representation of the line passing through A and B for*

1. $A = (1, 3, -1), \ \ B = (-4, 1, 2)$
2. $A = (-1, 5, 3), \ \ B = (-2, 4, 7)$

Exercise 1.15. *If P and Q are arbitrary points in $\mathbb{R}^n$, determine the general formula for the midpoint of the line segment between P and Q.*

Exercise 1.16. *Write the proofs for Properties 1.6.4.*

Exercise 1.17. *Determine the cosine of the angle between the two planes defined by*

$$\{(x,y,z) \in \mathbb{R}^3 \mid 2x - y + z = 0\} \quad \textit{and} \quad \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y - z = 1\}.$$

Same question for the planes defined by

$$\{(x,y,z) \in \mathbb{R}^3 \mid x = 1\} \quad \textit{and} \quad \{(x,y,z) \in \mathbb{R}^3 \mid 3x + 2y - 7z = 1\}.$$

Exercise 1.18. *Find the equation of the plane in $\mathbb{R}^3$ passing through the three points $P_1 = (1,2,-1)$, $P_2 = (-1,1,4)$ and $P_3 = (1,3,-2)$.*

Exercise 1.19. *Let $P = (-1,1,7)$, $Q = (1,3,5)$ and $N = (-1,1,-1)$. Determine the distance between the point Q and the plane $H_{P,N}$.*

Exercise 1.20. *Let $P = (1,1,1)$, $Q = (1,-1,2)$ and $N = (1,2,3)$. Find the intersection of the line passing through Q and having the direction N with the plane $H_{P,N}$.*

Exercise 1.21. *Determine the equation of the hyperplane in $\mathbb{R}^4$ passing through the point $(1,1,1,1)$ and which is parallel to the hyperplane defined by*

$$\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid 1\,x_1 + 2\,x_2 + 3\,x_3 + 4\,x_4 = 5\}.$$

Similarly, for any $n > 1$ determine the equation of the hyperplane in $\mathbb{R}^n$ passing through the point $(1,1,\dots,1)$ and which is parallel to the hyperplane defined by

$$\Big\{(x_1,x_2,\dots,x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} j\,x_j = n+1\Big\}.$$

Does something special happen for $n = 2$?