Report

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Spectral Theorem (bounded version)

\( \mathcal{H} \): Hilbert space

\( A : \mathcal{H} \to \mathcal{H} \) self-adjoint operator

\( U : \mathcal{H} \to \mathcal{H} \) unitary operator

Then there are spectral family \( \{ E_\lambda \} \), \( \{ F_\lambda \} \) such that

\[
A = \sum_{\lambda \in \sigma(A)} \lambda E_\lambda, \quad M = \sup \{ \langle Ax, x \rangle | \| x \| \leq 1 \},
\]

\[
m = \inf \{ \langle Ax, x \rangle | \| x \| \leq 1 \}.
\]

\[
U = \int_0^\infty e^{it} dF(t).
\]

Theorem

Let \( A \) be closed symmetric operator on \( \mathcal{H} \). The operator

\[
U : \mathcal{R}(A+iI) \to \mathcal{R}(A-iI)
\]

\[
(A+iI)x \mapsto (A-iI)x
\]

is isometry operator.

\( U \) is called a Cayley transform of \( A \).

Fact that

\( A \): self-adjoint \( \iff \) \( U \): unitary

holds.

Proof) Let \( x \) be an element in \( D(A) \),

\[
\| (A+iI)x \|^2 = \| Ax \|^2 + \| x \|^2 + 2\text{Im} \langle Ax, x \rangle
\]

\[
= \| Ax \|^2 + \| x \|^2
\]

\[
\| (A-iI)x \|^2 = \| Ax \|^2 - \| x \|^2.
\]

\[ \therefore U: \text{isometry operator} \]
Spectral Theorem

With any self-adjoint operator \( (A, D(A)) \) on a Hilbert space \( \mathcal{H} \) one can associate a unique spectral family \( \{E_\lambda\} \) such that

\[ D(A) = \text{Dom} \, A = \bigcup_{\lambda} \mathcal{D}(E_\lambda) \]

Proof) Let \( U \) be a Cayley transform of \( A \). Then \( U \) is unitary operator. \( U \) has a spectral decomposition

\[ U = \int_0^{2\pi} e^{i\varphi} \, dF(\varphi) \]

\( \{F(\varphi)\} \) is spectral family of \( U \) and

\[ F(0) = 0, \, F(2\pi - 0) = 1. \]

We define a complex function \( \zeta \) by

\[ \zeta(\varphi) = \frac{2e^{i\varphi} - 1}{2 + i} \quad (\varphi \in \mathbb{R}), \]

and real value \( \theta(\varphi) \) \( (\varphi \in \mathbb{R}) \) by \( \zeta(\varphi) = e^{i\theta(\varphi)} \). We defined a spectral family \( \{E_\lambda\} \) by

\[ E_\lambda = F(\theta(\lambda)) \quad (\lambda \in \mathbb{R}) \]

Suppose that \( A_0 = \int_0^\infty \lambda \, dE(\lambda) \). A Cayley transform of \( A_0 \) is

\[ U_0 = \int_0^\infty \frac{2e^{i\theta(\lambda)} - 1}{2 + i} \, dE(\lambda), \]

\[ U = \int_0^{2\pi} e^{i\varphi} \, dF(\varphi) = \int_0^{2\pi} e^{i\theta(\varphi)} \, dF(\varphi) = \int_0^{2\pi} e^{i\theta(\lambda)} \, dF(\theta(\lambda)) \]

\[ = \int_0^\infty \zeta(\lambda) \, dE(\lambda) = \int_0^\infty \frac{2e^{i\theta(\lambda)} - 1}{2 + i} \, dE(\lambda) = U_0 \]

This imply that Cayley transform of \( A \) and \( A_0 \) is equivalent. Thus

\[ A = A_0 \]

holds.

We show a uniqueness of spectral decomposition of \( A \). Suppose that

\[ A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda). \]

A Cayley transform of \( A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) \) is

\[ U = \int_{-\infty}^{\infty} \frac{2e^{i\theta(\lambda)} - 1}{2 + i} \, dE(\lambda). \]
\[ F(\theta) = \begin{cases} 
0 & (\theta \leq 0) \\
E'(i \frac{1+e^{i\theta}}{1-e^{i\theta}}) & (0 < \theta < 2\pi) \\
I & (2\pi \leq \theta).
\end{cases} \]

Then
\[ U = \sum_0 e^{i\alpha} dF'(\theta) \]
holds, and \( F'(\theta) = F(\theta) \) by the uniqueness of the spectral decomposition of unitary operator.

\[ \therefore E(\lambda) = E'(\lambda). \square \]
Spectral Theorem

Let \( H \) be a Hilbert space and \( \mathcal{A} \) be a bounded self-adjoint operator on \( H \). Then there is a spectral family \( \{ \mathcal{E}_\lambda \}_{\lambda \in \mathbb{R}} \) such that

\[
\mathcal{A} = \int_{-\infty}^{\infty} \lambda \, d\mathcal{E}_\lambda = \sum_{\lambda \in \mathbb{R}} \lambda \mathcal{E}_\lambda
\]

and each \( \mathcal{E}_\lambda \) is belong to the strong closure of

\[
P(\mathcal{A}) = \{ \sum_{n=0}^{\infty} \alpha_n \mathcal{A}^n | \{ \alpha_n \} \subseteq \mathbb{C} \}.
\]

I often see the proof of spectral theorem that constructs spectral projection \( \mathcal{E}_\lambda \) directly. But in this report, I adopt the way using the Gelfand-Naimark theorem in the theory of C*-algebra.

Namely:

Let \( \mathcal{A} \) be a abelian unital Banach \( * \)-algebra. We define

\[
\text{Sp}\mathcal{A} := \{ \phi : \mathcal{A} \to \mathbb{C} | \phi \text{ homomorphism}, \phi(0) = 0 \},
\]

\[
\wedge : \mathcal{A} \ni a \mapsto \hat{a} \in \text{C}(\text{Sp}\mathcal{A}),
\]

\[
\hat{a}(\phi) = \phi(a) \quad (\phi \in \text{Sp}\mathcal{A}).
\]

Then \( \mathcal{A} \) is a C*-algebra if and only if the map \( \wedge \) is an isometric \( * \)-isomorphism.

Proof of the spectral theorem) Let \( \mathcal{A} \) be a norm closure of \( P(\mathcal{A}) \). By the Gelfand-Naimark theorem, \( \mathcal{A} \cong \text{C}(\text{Sp}\mathcal{A}) \) holds. For each \( \lambda \in \mathbb{R} \), we define

\[
P_\lambda : \text{Sp}\mathcal{A} \equiv \{ \phi : \mathcal{A} \to \mathbb{C} | \phi(\mathcal{A}) \subseteq \mathbb{R} \},
\]

\[
K_\lambda = \{ \phi \in \text{Sp}\mathcal{A} | \hat{\lambda}(\phi) \leq 1 \}. \quad \text{(1)}
\]

We note that \( \hat{\lambda} \) is real function since \( \mathcal{A} \) is self-adjoint and \( \chi \) is a characteristic function. Since Urysohn's lemma, there is a \( \xi \in C(\text{Sp}\mathcal{A}) \) such that

\[
0 \leq \xi \leq 1,
\]

\[
\xi(\phi) = 1 \quad (\phi \in K_1),
\]

\[
\xi(\phi) = 0 \quad (\phi \notin K_1).
\]

For each \( x \in H \), \( \Phi : \mathcal{T} \to (H, \mathcal{A}) \) is a linear functional on \( C(\text{Sp}\mathcal{A}) \).
Thus by Riesz representation theorem, there is a regular Borel measure 
$\mu_x$ such that

$$
\Phi(T) = \langle T_x, x \rangle = \int S_{sp} \, \hat{T}(e) \, d\mu_x(e).
$$

There is a $T_x \in A$ s.t. $x = \hat{T}$, by $A \subseteq C(S_{sp})$. Since

$$
\|S^{n} - S^{n-1}\|_2 = \int \langle \overline{\hat{T}}(e) - \hat{T}(e) \rangle^n \, d\mu_x(e) = \|(T^n - T^{n-1}) x, x\|_2 = \|(T^n - T^{n-1}) e\|_2,
$$

each $\{T_x^n \}_{n \in \mathbb{N}}$ is Cauchy sequence in $C(A)$. Therefore $\lim_{n \to \infty} T^n_x = e$ exists.

It's clear that $E_x \in \mathcal{P}(A)$. A $E_x$ is independent of the choice of $x$ and $E_x = E_{x^*}$ (':' $x$ is real function),

$E_x^* = E_x$ (':' the definition of $E_x$).

Therefore $(E_x)_{x \in \mathbb{E}}$ is spectral family. For $\varepsilon > 0$, $n \in \mathbb{N}$, Suppose that

$$
-\|A\| - \varepsilon < \lambda_1 < \lambda_2 < \cdots < \lambda_{n+1} < \lambda_n = \|A\|.
$$

Then

$$
\chi_{\{\lambda \in \mathbb{C} : \delta < \lambda, |\lambda - \lambda_i| < \varepsilon \}}(e_{2i-1}) = P_{E_{1i}} - P_{E_{2i}}. \quad (\varepsilon \text{'s definition})
$$

Put

$$
S_x = \lim_{n \to \infty} \lambda_i (P_{2i-1} - P_{2i}).
$$

Since $S_x \to \hat{A}$ (uniformly), for any $\delta > 0$, there is $N \in \mathbb{N}$ such that

for all $n \geq N$ and any $x \in X$

$$
\|S_{sp} \, \lambda_i (P_{2i-1} - P_{2i}) - A \|_2 = \|S_{sp} \, S_{\delta \lambda} - \hat{A}(e) \|_2 \leq \delta S_{sp} \, d\mu_x(e) \leq \delta \|x\|_2.
$$

$$
\therefore \, A = S_{\|A\| + \varepsilon} \lambda \, dE_x \quad (\forall \varepsilon > 0).
$$

Since

$$
\lambda < \|A\| \Rightarrow P_\lambda = 0 \Rightarrow E_\lambda = 0,
$$

$$
\lambda > \|A\| \Rightarrow P_\lambda = I \Rightarrow E_\lambda = I,
$$

$$
A = \int \lambda \, dE_x = \int \lambda \, dE_x = \int \lambda \, dE_x
$$

holds. $\square$

Directly, a spectral family $(E_x)_{x \in \mathbb{E}}$ is defined by
\( E_x = \text{proj}(\{x \in \mathbb{C} \mid (A-\lambda 1)^t x = 0\}) \)
\( = \text{proj}(\{x \in \mathbb{C} \mid (A-\lambda 1)^- x = 0\}) \) (We suppose that \((A-\lambda 1)^t\) and \((A-\lambda 1)^-\) are positive part and negative part of \(A-\lambda 1\) respectively)

The spectral family \(\{E_x\}_{x \in \mathbb{R}}\) such that \(A = \int_{-\infty}^{\infty} \lambda dE_x\) is unique. Indeed, for \(\mu \in \mathbb{R}\), let \(\chi\) be the characteristic function of the interval \([\|A\|-1, \|A\|]\) and \(\{p_n\}\) be a sequence of polynomials converging to \(\chi\). Then for \(x \in \mathbb{C}\),
\[
\|E_x x\| = (E_x x, x) \\
= \int \chi(x) \alpha(E_x x) \\
= \lim_{n \to \infty} \int p_n(x) \alpha(E_x x) (\because \text{Lebesgue's dominated convergence theorem}) \\
= \lim_{n \to \infty} \int p_n(x) \alpha x (\because p_n \text{ is polynomial}).
\]

Since \(p_n\) is independent of \(E_x\), if we assume that the spectral family \(\{F_x\}\) satisfies \(A = \int_{-\infty}^{\infty} \lambda dF_x\), then \(E_x = F_x\) (\(x \in \mathbb{R}\)).

\(<\text{Comments}>\)
I had left out the proof of that the \(\{E_x\}_{x \in \mathbb{R}}\) defined by two ways is spectral family.