Report

321401115
Yusuke Sawada

Spectral Theorem (bounded version)

$\mathcal{H}$: Hilbert space

$A : \mathcal{H} \to \mathcal{H}$ self-adjoint operator

$U : \mathcal{H} \to \mathcal{H}$ unitary operator

Then there are spectral family $\{E_\lambda\}, \{F_\theta\}$ such that

$$A = \int_{m-0}^{M} \lambda \, dE(\lambda), \quad M = \sup\{(Ax, x) \mid \|x\| \leq 1\}, \quad m = \inf\{(Ax, x) \mid \|x\| \leq 1\},$$

$$U = \int_0^{2\pi} e^{i\theta} \, dF(\theta).$$

Theorem

Let A be closed symmetric operator on $\mathcal{H}$. The operator

$$\begin{array}{ccc} U : R(A+iI) & \longrightarrow & R(A-iI) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (A+iI)x & \longmapsto & (A-iI)x \end{array}$$

is isometry operator.

U is called a Cayley transform of A.

Fact that

$$A : \text{self-adjoint} \Leftrightarrow U : \text{unitary}$$

holds.

Proof) Let x be a element in $D(A)$.

$$\begin{aligned} \|(A+iI)x\|^2 &= \|Ax\|^2 + \|x\|^2 + 2\,\mathrm{Im}(Ax, x) \\ &= \|Ax\|^2 + \|x\|^2, \end{aligned}$$

$$\|(A-iI)x\|^2 = \|Ax\|^2 + \|x\|^2.$$

$\therefore U$: isometry operator $\square$

Spectral Theorem

With any self-adjoint operator $(A, D(A))$ on a Hilbert space $\mathcal{H}$ one can associate a unique spectral family $\{E_\lambda\}$ such that

$$D(A) = D_{id}, \quad A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda).$$

Proof) Let U be a Cayley transform of A. Then U is unitary operator. U has a spectral decomposition

$$U = \int_0^{2\pi} e^{i\theta} dF(\theta)$$

$\{F(\theta)\}$ is spectral family of U and

$$F(0) = O, \; F(2\pi - 0) = I.$$

We define a complex function ξ by

$$\xi(\lambda) = \frac{\lambda - i}{\lambda + i} \quad (\lambda \in \mathbb{R}),$$

and real value $\theta(\lambda)$ $(\lambda \in \mathbb{R})$ by $\xi(\lambda) = e^{i\theta(\lambda)}$. We defined a spectral family $\{E_\lambda\}$ by

$$E_\lambda = F(\theta(\lambda)) \quad (\lambda \in \mathbb{R})$$

Suppose that $A_0 = \int_{-\infty}^{\infty} \lambda \, dE(\lambda)$. A Cayley transform of A_0 is

$$U_0 = \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda + i} dE(\lambda).$$

$$U = \int_0^{2\pi} e^{i\theta} dF(\theta) = \int_{0+0}^{2\pi - 0} e^{i\theta} dF(\theta) = \int_{-\infty}^{\infty} e^{i\theta(\lambda)} dF(\theta(\lambda))$$

$$= \int_{-\infty}^{\infty} \xi(\lambda) dE(\lambda) = \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda + i} dE(\lambda) = U_0.$$

This imply that Cayley transform of A and A_0 is equivalent. Thus $A = A_0$ holds.

We show a uniqueness of spectral decomposition of A. Suppose that

$$A = \int_{-\infty}^{\infty} \lambda \, dE'(\lambda).$$

A Cayley transform of $A = \int_{-\infty}^{\infty} \lambda \, dE'(\lambda)$ is

$$U = \int_{-\infty}^{\infty} \frac{\lambda - i}{\lambda + i} dE'(\lambda).$$

Put

$$F'(\theta)=\begin{cases} 0 & (\theta \leqq 0) \\ E'\left(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\right) & (0<\theta<2\pi) \\ I & (2\pi \leqq \theta), \end{cases}$$

Then

$$U=\int_0^{2\pi} e^{i\theta}\,dF'(\theta)$$

holds, and $F'(\theta)=F(\theta)$ by the uniqueness of the spectral decomposition of unitary operator.

$$\therefore E(\lambda)=E'(\lambda). \quad \square$$

~Reference~

函数解析 1968 竹之内脩

32140115 Yusuke Sawada

Spectral Theorem

Let $\mathcal{H}$ be a Hilbert space and A be a bounded self-adjoint operator on $\mathcal{H}$. Then there is a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ such that

$$A=\int_{-\infty}^{\infty}\lambda\, dE_\lambda=\lim_{\varepsilon\to+0}\int_{-\|A\|-\varepsilon}^{\|A\|}\lambda\, dE_\lambda$$

and each E_λ is belong to the strong closure of

$$P(A)=\{\sum_{i=0}^{n}a_iA^i \mid n\in\mathbb{N},\ \{a_i\}_{i=0}^{n}\subset\mathbb{C}\}.$$

I often see the proof of spectral theorem that constructs spectral projection E_λ directly. But in this report, I adopt the way using the Gelfand-Naimark theorem in the theory of C^*-algebra.

Namely:

Let $\mathcal{A}$ be a abelian unitial Banach $*$-algebra. We define

$$Sp\mathcal{A}:=\{\varphi:\mathcal{A}\to\mathbb{C}\mid \varphi:\text{homomorphism},\ \varphi\neq 0\},$$

$$\hat{}:\mathcal{A}\ni a\longmapsto \hat{a}\in C(Sp\mathcal{A}),$$

$$\hat{a}(\varphi)=\varphi(a)\quad(\varphi\in Sp\mathcal{A}).$$

Then $\mathcal{A}$ is a C^*-algebra if and only if the map $\hat{}$ is an isometric $*$-isomorphism.

Proof of the spectral theorem) Let $\mathcal{A}$ be a norm closure of $P(A)$. By the Gelfand-Naimark theorem, $\mathcal{A}\cong C(Sp\mathcal{A})$ holds. For each $\lambda\in\mathbb{R}$, we define

$$p_\lambda: Sp\mathcal{A}\ni\varphi\longmapsto \chi_{K_\lambda}(\varphi),\quad K_\lambda=\{\varphi\in Sp\mathcal{A}\mid \hat{A}(\varphi)\leq\lambda\}.\quad \cdots ①$$

We note that $\hat{A}$ is real function since A is self-adjoint and χ is a characteristic function. Since Urysohn's lemma, there is a $\xi\in C(Sp\mathcal{A})$ such that

$$0\leq\xi\leq 1,$$

$$\xi(\varphi)=1\quad(\varphi\in K_\lambda),$$

$$\cancel{\xi(\psi)=0\quad(\psi\notin K_\lambda)}$$

For each $x\in\mathcal{H}$, $\underset{\substack{\uparrow\\ C(Sp\mathcal{A})}}{\Phi:\hat{T}}\longmapsto \underset{\substack{\uparrow\\ \mathbb{C}}}{(Tx,x)}$ is a linear functional on $C(Sp\mathcal{A})$.

32140115 Yusuke Sawada

Thus by Riesz representation theorem, there is a regular Borel measure μ_x such that

$$\Phi(T) = (Tx, x) = \int_{Sp\mathcal{A}} \hat{T}(\varphi)\, d\mu_x(\varphi).$$

There is a $T_\xi \in \mathcal{A}$ s.t. $\xi = \hat{T}_\xi$ by $\mathcal{A} \cong C(Sp\mathcal{A})$. Since

$$\|\xi^n - \xi^m\|_{L^2}^2 = \int_{Sp\mathcal{A}} |\hat{T}^n(\varphi) - \hat{T}^m(\varphi)|^2 d\mu_x(\varphi) = ((T^n - T^m)^*(T^n - T^m)x, x) = \|(T^n - T^m)x\|^2,$$

each $\{T_\xi^n x\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{H}$. Therefore $s\text{-}\lim_{n\to\infty} T_\xi^n =: E_\lambda$ exists.

It's clear that $E_\lambda \in \overline{P(\mathcal{A})}^s$. A E_λ is independent of the choice of ξ and

$$E_\lambda = E_\lambda^* \quad (\because \xi \text{ is real function}),$$

$$E_\lambda^2 = E_\lambda \quad (\because \text{the definition of } E_\lambda)$$

hold. Therefore $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is spectral family. For $\varepsilon > 0, n \in \mathbb{N}$, Suppose that

$$-\|A\| - \varepsilon = \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n = \|A\|.$$

Then

$$\chi_{\{\varphi\in Sp\mathcal{A} \mid \lambda_i < \hat{A}(\varphi) \leq \lambda_{i+1}\}} = P_{\lambda_{i+1}} - P_{\lambda_i}. \quad (① \text{'s definition})$$

Put

$$S_n = \sum_{i=1}^{n} \lambda_i (P_{\lambda_{i+1}} - P_{\lambda_i}).$$

Since $S_n \to \hat{A}$ (uniformly), for any $\delta > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ and any $x \in \mathcal{H}$

$$\begin{aligned}\left\|\left(\sum_{i=1}^{n} \lambda_{i+1}(E_{\lambda_{i+1}} - E_{\lambda_i}) - A\right)x\right\|^2 &= \int_{Sp\mathcal{A}} |S_n(\varphi) - \hat{A}(\varphi)|^2 d\mu_x(\varphi) \\ &< \delta \int_{Sp\mathcal{A}} d\mu_x(\varphi) \\ &= \delta\|x\|^2.\end{aligned}$$

$$\therefore A = \int_{-\|A\|-\varepsilon}^{\|A\|} \lambda\, dE_\lambda \quad (\forall \varepsilon > 0).$$

Since

$$\lambda < -\|A\| \Rightarrow P_\lambda = 0 \Rightarrow E_\lambda = O,$$

$$\lambda > \|A\| \Rightarrow P_\lambda = 1 \Rightarrow E_\lambda = I$$

$$A = \int_{-\infty}^{\infty} \lambda\, dE_\lambda = \lim_{\varepsilon\to+0} \int_{-\|A\|-\varepsilon}^{\|A\|} \lambda\, dE_\lambda$$

holds. $\square$

Directly, a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is defined by

321401115 Yusuke Sawada

$$E_\lambda = \mathrm{proj}(\{x \in \mathcal{H} \mid (A-\lambda I)^+ x = 0\})$$
$$= \mathrm{proj}(\{x \in \mathcal{H} \mid (A-\lambda I)^- x = 0\}).$$

(We suppose that $(A-\lambda I)^+$ and $(A-\lambda I)^-$ are positive part and negative part of $A-\lambda I$ respectively)

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ such that
$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda,$$
is unique. Indeed, for $\mu \in \mathbb{R}$, let g be the characteristic function of the interval $(-\|A\|-1, \mu]$ and $\{p_n\}$ be a sequence of polynomials converging to g. Then for $x \in \mathcal{H}$,
$$\|E_\mu x\|^2 = (E_\mu x, x)$$
$$= \int g(\lambda)\, d(E_\lambda x, x)$$
$$= \lim_{n\to\infty} \int p_n(\lambda)\, d(E_\lambda x, x) \quad (\because \text{Lebesgue's dominated convergence theorem})$$
$$= \lim_{n\to\infty} (p_n(A)x, x) \quad (\because p_n \text{ is polynomial}).$$
Since p_n is independent of E_μ, if we assume that the spectral family $\{F_\lambda\}_\lambda$ satisfies $A = \int_{-\infty}^{\infty} \lambda \, dF_\lambda$, then $E_\lambda = F_\lambda$ $(\lambda \in \mathbb{R})$.

<Coments>

I had left out the proof of that the $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ defined by two ways is spectral family.