C^*-algebraic methods in spectral theory, Extension 2.5.12.

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GNS representation

Definition 1. Let A be a unitial C^*-algebra. *A positive linear functional* on A is a linear map $\omega : A \to \mathbb{C}$, such that $\omega(a^*a) \geq 0\,(\forall a \in A)$. Moreover, a positive linear functional ω such that $\omega(1) = 1$ is called *a state* on A.

Definition 2. Let A be a unitial C^*-algebra and (H, π) be a representation of A. A vector $\Omega \in H$ is called *a cyclic vector* for the representation if $\pi(A)\Omega := \{\pi(a)\Omega : a \in A\}$ is dense in H. If (H, π) has a cylic vector, it is called *a cyclic representation*.

Proposition 1. Let ω be a state on a C^*-algebra A. For $a \in A$ such that $a = a^*$, $\omega(a) \in \mathbb{R}$. In addition, we have the inequality $|\omega(b^*a)| \leq \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}\,(\forall a,\, b \in A)$.

Proof. Let $a = a^* \in A$. Since ω is a state, we have $0 \leq \omega((1+a)^*(1+a)) = \omega(1^*1+2a+a^*a) = \omega(1^*1) + 2\omega(a) + \omega(a^*a)$. Since $\omega(1^*1) \geq 0$, $\omega(a^*a) \geq 0$, we have $\omega(a) \in \mathbb{R}$.
For $a,\, b \in A$ we define $\langle a, b\rangle := \omega(b^*a)$. Then $\langle \cdot, \cdot \rangle$ is a semi-inner product in A. So the inequality follows from Cauchy-Schwarz inequality in semi-inner product spaces. $\square$

Theorem 1. (GNS representation with respect to a state) Let A be a unitial C^*-algebra and ω be a state on A. Then there is a cyclic representation (H, π, Ω) of A, with $\langle \pi(a)\Omega, \Omega\rangle = \omega(a)$.

Proof. We set $N := \{a \in A : \omega(a^*a) = 0\}$. Then, note that N is a left ideal in A. Indeed we have

$$\begin{aligned} a \in A,\, x \in N \Rightarrow \omega((ax)^*ax) = \omega(x^*a^*ax) \leq \omega(x^*x)^{1/2}\omega((a^*ax)^*(a^*ax))^{1/2} = 0 \\ \Rightarrow \omega((ax)^*ax) = 0 \Rightarrow ax \in N. \end{aligned}$$

For $a \in A$, we set $[a] := \{x \in A : x - a \in N\}$. For $[x]$, $[y] \in A/N$ we define $\langle [x], [y]\rangle := \omega(y^*x)$. When $[x_1] = [x_2]$ and $[y_1] = [y_2]$, we have

$$\begin{aligned} |\langle [x_1], [y_1]\rangle - \langle [x_2], [y_2]\rangle| &= |\omega({y_1}^*x_1) - \omega({y_2}^*x_2)| \\ &\leq |\omega((y_1 - y_2)^*x_1)| + |\omega({y_2}^*(x_1 - x_2))| \\ &\leq \omega((y_1 - y_2)^*(y_1 - y_2))^{1/2}\omega({x_1}^*x_1)^{1/2} + \omega({y_2}^*y_2)^{1/2}\omega((x_1 - x_2)^*(x_1 - x_2))^{1/2} \\ &= 0. \end{aligned}$$

So $\langle \cdot, \cdot \rangle$ is a well-defined sesquilinear form on A/N. If $\langle [x], [x]\rangle = 0$, then $\omega(x^*x) = 0$, and thus $[x] = [0]$. Therefore $\langle \cdot, \cdot \rangle$ is an inner product on A/N. Moreover, $||[x]||_\omega := \langle [x], [x]\rangle$ is a norm on A/N. For $a \in A$, we define an action L_a of A on A/N by $L_a[x] = [ax]\,([x] \in A/N)$.
Since N is a left ideal of A, we have

$$[x] = [y] \Rightarrow x - y \in N \Rightarrow a(x - y) \in N \Rightarrow ax - ay \in N \Rightarrow [ax] = [ay].$$

So the action is well-defined. Moreover, we get

$$\begin{aligned} ||L_a[x]||_\omega^2 &= \langle L_a[x], L_a[x]\rangle \\ &= \omega((ax)^*ax) \\ &= \omega(x^*a^*ax). \end{aligned}$$

We set $\phi(b) := \omega(x^*bx)$ for all $b \in A$. Then ϕ is a positive linear functional on A and so $|\phi(b)| \leq \phi(1)||b||\,(\forall b \in A)$. Therefore $|\omega(x^*bx)| \leq \omega(x^*x)||b||$. In particular for $b = a^*a$, we have

$$\begin{aligned} |\omega(x^*a^*ax)| &\leq \omega(x^*x)||a^*a|| \\ &= \omega(x^*x)||a||^2 \\ &= \langle [x], [x]\rangle\, ||a||^2 \\ &= ||[x]||_\omega\, ||a||^2. \end{aligned}$$

Hence we have $||L_a[x]||_\omega \leq ||a||\, ||[x]||_\omega$, and it follows that $L_a \in B(A/N)$.
In addition, let us observe that :

$$\begin{aligned} &L_{a+b}[x] = [(a+b)x] = [ax] + [bx] = L_a[x] + L_b[x] = (L_a + L_b)[x], \\ &L_{ab}[x] = [(ab)x] = [a(bx)] = L_a[bx] = L_aL_b[x], \\ &L_1[x] = [x] = 1[x]. \end{aligned}$$

Thus

$$\begin{aligned} &L_{a+b} = L_a + L_b, \\ &L_{ab} = L_aL_b, \\ &L_1 = 1, \end{aligned}$$

and we also get :

$$\langle L_{a^*}[x], [y]\rangle = \langle [a^*x], [y]\rangle = \omega(y^*a^*x) = \omega((ay)^*x) = \langle [x], [ay]\rangle = \langle [x], L_a[y]\rangle\,,$$

from which we have

$$L_a{}^* = L_{a^*}.$$

Let H be the completion of A/N with the norm $||\cdot||_\omega$. Then H is a Hilbert space and A/N is dense in H. We set $\Omega := [1] \in A/N$. Then $[a] = [a1] = L_a[1] = L_a\Omega\,(\forall [a] \in A/N)$. So, $A/N = \{L_a\Omega : a \in A\}$. Since L_a is bounded on A/N for all $a \in A$, they have a unique continuous extension, $\pi(a) \in B(H)$. Ω is a cyclic vector for the representation (H, π) since $A/N = \{L_a\Omega : a \in A\} = \{\pi(a)\Omega : a \in A\} = \pi(A)\Omega$ is dense in H.
Finally, we get $\langle \pi(a)\Omega, \Omega\rangle = \langle L_a[1], [1]\rangle = \langle [a], [1]\rangle = \omega(a)$. □

Theorem 2. Let A be a unitial C^*-algebra. Then the set of states on A separates the point of A.

Proof. For $a, b \in A\,(a \neq b)$, we set $x := a - b$ and write $x = j + ik\,(j = j^*,\, k = k^*)$. Since $x \neq 0$, $j \neq 0$ or $k \neq 0$. If ω is a state, then $\omega(j)$, $\omega(k) \in \mathbb{R}$. So $\omega(x) \neq 0$ if and only if $\omega(j) \neq 0$ or $\omega(k) \neq 0$. So we can show the theorem if we can prove the following statements :
For any $x = x^*\,(x \neq 0)$, there is a state ω such that $\omega(x) \neq 0$.
To prove this we use the identification $C^*(x) :=$ (the untial C^*-algebra generated by x) $\simeq C(Sp\, C^*(x))$. Since $x \neq 0$ there is $\chi \in Sp\, C^*(x)$ such that $\hat{x}(\chi) \neq 0$ ($\hat{}$ is the Gelfand Transform). We define $\omega_0 : C^*(x) \to \mathbb{C}$ by $\omega_0(a) := \hat{a}(\chi)\,(\forall a \in C^*(x))$. By Hahn-Banach theorem, there is a continuous linear extension of ω_0 on A. Let ω be the extension of ω_0. Then ω is a state on A and we get $\omega(x) = \omega_0(x) = \hat{x}(\chi) \neq 0$. □

Theorem 3. For any C^*-algebra A, there exist a faithful representation.

Proof. Without loss of generality, we can suppose that A is unitial. If not, we consider the unitialization of A. We set $S(A) :=$ (the set of the states on A), and for each $\omega \in S(A)$, let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS representation of A with respect to the state ω. We set $H := \bigoplus_{\omega \in S(A)} H_\omega$, $\pi(a) := \bigoplus_{\omega \in S(A)} \pi_\omega(a)\,(\forall a \in A)$ and $\Omega := \bigoplus_{\omega \in S(A)} \Omega_\omega$. Then (H, π, Ω) is the GNS representation of A.
If $\pi(a) = \pi(b)$, then $0 = (\pi(a) - \pi(b))\Omega = \bigoplus_{\omega \in S(A)}(\pi_\omega(a) - \pi_\omega(b))\Omega_\omega$. Hence $(\pi_\omega(a) - \pi_\omega(b))\Omega_\omega = 0\,(\forall \omega \in S(A))$. Therefore $0 = \langle(\pi_\omega(a) - \pi_\omega(b))\Omega_\omega, \Omega_\omega\rangle = \omega(a - b)\,(\forall \omega \in S(A))$. Since $S(A)$ separates the point of A, we get $a = b$. Therefore π is faithful. □

References

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