The term paper of the lecture on C^*-algebraic methods in spectral theory

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Exercise 1.3.3 *For the operator A_n defined in* (1.3.1)*, give an upper estimate for $\|A_n\|$ and compute A_n^*.*

By definition of norm, we can give an upper estimate

$$\begin{aligned}\|A_n\| = \sup_{\|f\|=\|g\|=1} |\langle A_n f, g\rangle| &= \sup_{\|f\|=\|g\|=1} \left|\sum_{j=1}^{n} \langle f, g_j\rangle \langle h_j, g\rangle\right| \\ &\leq \sup_{\|f\|=\|g\|=1} \sum_{j=1}^{n} |\langle f, g_j\rangle| |\langle h_j, g\rangle| \leq \sum_{j=1}^{n} \|g_j\| \|h_j\|.\end{aligned}$$

For its adjoint, one has :

$$\langle A_n f, g\rangle = \left\langle \sum_{j=1}^{n} \langle f, g_j\rangle h_j, g \right\rangle = \sum_{j=1}^{n} \langle f, g_j\rangle \langle h_j, g\rangle = \left\langle f, \sum_{j=1}^{n} \overline{\langle h_j, g\rangle} g_j \right\rangle$$

for any $f, g \in \mathcal{H}$. Therefore

$$A_n^* g = \sum_{j=1}^{n} \overline{\langle h_j, g\rangle} g_j = \sum_{j=1}^{n} \langle g, h_j\rangle g_j$$

for any $g \in \mathcal{H}$.

Exercise 2.4.16 *Let $\mathbf{\Omega}$ be a compact Hausdorff space, and for each $x \in \mathbf{\Omega}$ let τ_x be the character on $C(\mathbf{\Omega})$ defined by $\tau_x(f) = f(x)$ for any $f \in C(\mathbf{\Omega})$. Show that the map*

$$\mathbf{\Omega} \ni x \mapsto \tau_x \in \Omega(C(\mathbf{\Omega}))$$

is a homeomorphism.

Since the map $\varphi : \mathbf{\Omega} \ni x \mapsto \tau_x \in \Omega(C(\mathbf{\Omega}))$ is a map from Hausdorff space to compact space, if φ is bijective and continuous then this map is a homeomorphism.

Since $\mathbf{\Omega}$ is a compact Hausdorff space, this space is normal. Therefore for any two different points $x, y \in \mathbf{\Omega}$ there exist two open subsets U, V on $\mathbf{\Omega}$ which satisfy $x \in U$, $\overline{U} \subset V$ and $y \notin V$. By using Urysohn's lemma, there exists a continuous map $f : \mathbf{\Omega} \to \mathbb{C}$ such that $f \equiv 1$ on U and $f \equiv 0$ on $\mathbf{\Omega} \setminus V$. Then $\tau_x(f) = f(x) = 1$ and $\tau_y(f) = f(y) = 0$, so $\tau_x \neq \tau_y$. By contraposition, if $\tau_x = \tau_y$ then $x = y$, that is φ is injective.

By using a proof by contradiction, we will prove the map φ to be surjective. Assume that φ is not surjective then there exists $\tau \in \Omega(C(\mathbf{\Omega}))$ which satisfies that for each $x \in \mathbf{\Omega}$ there exists $f \in C(\mathbf{\Omega})$ such that $\tau(f) \neq f(x)$. Because of $\dim C(\mathbf{\Omega}) \geq \dim \mathbb{C}$, $\ker \tau = \{f \in C(\mathbf{\Omega}) \mid \tau(f) = 0\}$ is not empty. Hence for each $x \in \mathbf{\Omega}$ there exists $f_x \in \ker \tau$ such that $f_x(x) \neq 0$. Since $\ker \tau$ is an ideal on the C^*-algebra $C(\mathbf{\Omega})$ and τ is a C^*-homomorphism, then

$$\tau(\overline{f}_x) = \tau(f_x)^* = 0^* = 0.$$

Therefore $\overline{f}_x \in \ker \tau$ and $|f_x|^2 = \overline{f}_x f_x \in \ker \tau$.

Let $\{U_x\}$ be an open covering of $\mathbf{\Omega}$ such that U_x is a neighborhood of x on $\mathbf{\Omega}$ and $|f_x|^2$ is positive function on U_x. Because of the compactness of $\mathbf{\Omega}$, there exists a finite subcovering $\{U_{x_i}\}_{i=1}^n$. Setting $f := |f_{x_1}|^2 + \cdots + |f_{x_n}|^2$. Then f is finite summation of non-negative functions $|f_{x_1}|^2, \ldots, |f_{x_n}|^2$ and for each $x \in \mathbf{\Omega}$ there exists U_{x_j} and $|f_{x_j}|^2$ satisfying $x \in U_{x_j}$ and $f_{x_j}(x) > 0$. Therefore f is non-zero function anywhere. In addition, since $\tau(f) = \tau(|f_{x_1}|^2) + \cdots + \tau(|f_{x_n}|^2) = 0$, $f \in \ker \tau$.

Since f is non-zero anywhere, $1/f \in C(\mathbf{\Omega})$ is well-defined. Hence

$$1 = f\frac{1}{f} \in \ker \tau,$$

thus $\ker \tau = C(\mathbf{\Omega})$. However τ is not zero, so this is contradiction. Therefore φ is surjective.

Lastly, we will prove φ to be continuous. Let U be an open set on $\Omega(C(\mathbf{\Omega}))$ and let $x \in \varphi^{-1}(U)$. Then $\tau_x \in U$. Since U is open, there exist $\epsilon > 0$ and a finite subset $S \subset C(\mathbf{\Omega})$ such that

$$N(\tau_x : \epsilon, S) := \{\tau \in \Omega(C(\mathbf{\Omega})) \mid |\tau(f) - \tau_x(f)| < \epsilon, \forall f \in S\} \subset U.$$

Since f is continuous at x for each $f \in S$, there exists an open neighborhood V_f of x on $\mathbf{\Omega}$ satisfying $|f(y) - f(x)| < \epsilon$ for any $y \in V_f$. Let $V = \bigcap_{f \in S} V_f$. Since S is a finite set, V is an open set on $\mathbf{\Omega}$ and $x \in V$. In addition, for each $y \in V$, one has

$$|f(y) - f(x)| < \epsilon$$

for any $f \in S$. Equivalently, for each $y \in V$, that means

$$|\tau_y(f) - \tau_x(f)| < \epsilon$$

for any $f \in S$. Thus $\tau(V) \subset N(\tau_x : \epsilon, S)$. Therefore $x \in V \subset \tau^{-1}(N(\tau_x : \epsilon, S))$ and it follows that φ is continuous.

Exercise 3.1.12 *We state in this exercise a couple of useful formulas which can be deduced from the definition of the modular function. Let $f \in C_c(G)$ and $x \in G$:*

$$\int_G f(xy)\, d\mu(y) = \int_G f(y)\, d\mu(y),$$
$$\int_G f(yx)\, d\mu(y) = \Delta(x)^{-1} \int_G f(y)\, d\mu(y),$$
$$\int_G \Delta(y^{-1}) f(y^{-1})\, d\mu(y) = \int_G f(y)\, d\mu(y).$$

First, since μ is a left Haar measure, it satisfies $\mu(xV) = \mu(V)$. Let $y' = xy$. Then we have

$$\int_G f(xy)\, d\mu(y) = \int_G f(xy)\, d\mu(xy) = \int_G f(y')\, d\mu(y') = \int_G f(y)\, d\mu(y).$$

Second, put $g_x(y) = [R_x f](y) = f(yx)$ for any $x \in G$. Now, by lemma 3.1.7,

$$\int_G f(y)\, d\mu(y) = \int_G [R_{x^{-1}} g_x](y)\, d\mu(y) = \Delta(x) \int_G g_x(y)\, d\mu(y) = \Delta(x) \int_G f(yx)\, d\mu(y),$$

so

$$\int_G f(yx)\, d\mu(y) = \Delta(x)^{-1} \int_G f(y)\, d\mu(y).$$

Third, we define the new measure ν as

$$\int_G f(y)\, d\nu(y) := \int_G f(y^{-1}) \Delta(y^{-1})\, d\mu(y)$$

for any $f \in C_c(G)$. Since we have

$$\begin{aligned}\int_G f(x^{-1}y)\, d\nu(y) = &\int_G f(x^{-1}y^{-1}) \Delta(y^{-1})\, d\mu(y) \\ =&\Delta(x) \int_G f((yx)^{-1}) \Delta((yx)^{-1})\, d\mu(y) \\ =&\int_G f(y^{-1}) \Delta(y^{-1})\, d\mu(y) = \int_G f(y)\, d\nu(y)\end{aligned}$$

for any positive function $f \in C_c(G)$ and $x \in G$ by the second formula, ν is a left Haar measure. By the uniqueness of a Haar measure, ν equals to μ up to a positive scalar c. Now, let $g \in C_c(G)$ be a positive function, and define h as $h(y) := g(y) + \Delta(y^{-1}) g(y^{-1})$. For this function h, we get $h(y) = \Delta(y^{-1}) h(y^{-1})$ and so

$$\int_G h(y)\, d\mu(y) = \int_G h(y)\, d\nu(y).$$

This implies $c = 1$.

Exercise 4.3.8 *In the framework of the previous paragraph, show that* $\sigma(\Phi^H) = \sigma(H)$.

Suppose $\Phi^H(\varphi) = \varphi(H) = 0$. According to Prop.1.7.11, we have

$$0 = \|\varphi(H)\| = \sup_{\lambda \in \sigma(H)} |\varphi(\lambda)| \geq 0.$$

Therefore for any $\lambda \in \sigma(H)$ we get $\varphi(\lambda) = 0$. By contraposition, it follows that if $\varphi \in C_0(\mathbb{R})$ satisfies $\varphi(\lambda) \neq 0$ for $\lambda \in \sigma(H)$ then $\Phi^H(\varphi) \neq 0$, namely if $\lambda \in \sigma(H)$ then $\lambda \in \sigma(\Phi^H)$.

Conversely, let $\lambda \in \mathbb{R} \setminus \sigma(H)$, and let $\{E_\rho\}$ be the spectral family corresponding to the self-adjoint operator H. Since we have

$$\sigma(H) = \text{Supp}\{E_\mu\} = \{\mu \in \mathbb{R} \mid E_{\mu+\epsilon} - E_{\mu-\epsilon} \neq 0, \forall \epsilon > 0\},$$

there exists $\epsilon > 0$ such that $E_{\lambda+\epsilon} - E_{\lambda-\epsilon} = 0$. Thus $(\lambda-\epsilon, \lambda+\epsilon] \subset \mathbb{R}$ is a zero spectral measure associated with the family $\{E_\rho\}$. Since this interval has a none zero Lebesgue measure, we can construct non-negative function $\varphi \in C_0(\mathbb{R})$ satisfies $\varphi(\lambda) \neq 0$ and $\text{Supp}\varphi \subset (\lambda-\epsilon, \lambda+\epsilon]$. Then φ is almost everywhere zero about the spectral measure. Therefore we get

$$\Phi^H(\varphi) = \varphi(H) = 0.$$

Hence there exists $\varphi \in C_0(\mathbb{R})$ such that $\varphi(\lambda) \neq 0$ and $\varphi(H) = 0$, so that $\lambda \notin \sigma(\Phi^H)$.

Exercise 5.1.5 *Check carefully the statements contained in the previous example.*

Let $\tilde{\mathcal{H}} = L^2(G; \mathcal{H})$. First, we will check that $(\tilde{\mathcal{H}}, \tilde{\pi})$ is a representation of $\mathscr{S}$. $\tilde{\mathcal{H}}$ is a Hilbert space. Let α, $\beta \in \mathbb{C}$ and let φ, $\psi \in \mathscr{S}$. Then for any $h \in \tilde{\mathcal{H}}$ and for any $x \in G$,

$$\begin{aligned}
[\tilde{\pi}(\alpha\varphi + \beta\psi)h](x) =& \pi(\theta_x(\alpha\varphi + \beta\psi))h(x) \\
=& \alpha\pi(\theta_x(\varphi))h(x) + \beta\pi(\theta_x(\psi))h(x) = [(\alpha\tilde{\pi}(\varphi) + \beta\tilde{\pi}(\psi))h](x), \\
[\tilde{\pi}(\varphi\psi)h](x) =& \pi(\theta_x(\varphi\psi))(x) = \pi(\theta_x\varphi)\pi(\theta_x\psi)h(x) = [\tilde{\pi}(\varphi)\tilde{\pi}(\psi)h](x),
\end{aligned}$$

and

$$[\tilde{\pi}(\varphi^*)h](x) = \pi(\theta_x(\varphi^*))h(x) = \pi(\theta_x\varphi)^*h(x) = [\tilde{\pi}(\varphi)^*h](x),$$

since π and θ_x are $*$-homomorphism. Therefore $\tilde{\pi}$ is also $*$-homomorphism and $(\tilde{\mathcal{H}}, \tilde{\pi})$ is a representation of $\mathscr{S}$.

Second, we will check that $\tilde{U}$ satisfies the condition (5.1.3). For any $h \in \tilde{\mathcal{H}}$ and for any $x, y, z \in G$, one has

$$\begin{aligned}[\tilde{U}_y \tilde{U}_z h](x) = &\left[\tilde{U}_y[\pi(\omega(\cdot, z))h(\cdot + z)]\right](x) = \pi(\omega(x,y))\pi(\omega(x+y,z))h(x+y+z)\\ =&\pi(\theta_x[\omega(y,z)]\omega(x,y+z))h(x+y+z) = [\tilde{\pi}(\omega(x,y))\tilde{U}_{y+z}h](x).\end{aligned}$$

Therefore $\tilde{U}$ satisfies the condition.

Last, we check that $\tilde{\pi}$ and $\tilde{U}$ satisfy the compatibility condition.

For any $y \in G$ and $h, g \in \tilde{\mathcal{H}}$, one has

$$\begin{aligned}\left\langle \tilde{U}_y^* h, g\right\rangle_{\tilde{\mathcal{H}}} =&\left\langle h, \tilde{U}_y g\right\rangle_{\tilde{\mathcal{H}}} = \int_G \left\langle h(x), [\tilde{U}_y g](x)\right\rangle_{\mathcal{H}} \, dx\\ =&\int_G \langle h(x), \pi(\omega(x,y))g(x+y)\rangle_{\mathcal{H}} \, dx\\ =&\int_G \langle h(z-y), \pi(\omega(z-y,y))g(z)\rangle_{\mathcal{H}} \, dz\\ =&\int_G \langle \pi(\omega(z-y,y)^*)h(z-y), g(z)\rangle_{\mathcal{H}} \, dz = \langle \pi(\omega(\cdot - y, y)^*)h(\cdot - y), g\rangle_{\tilde{\mathcal{H}}}.\end{aligned}$$

Thus we get $[\tilde{U}_y^* h](x) = \pi(\omega(x-y,y)^*)h(x-y)$. Then we have

$$\begin{aligned}[\tilde{U}_y \tilde{\pi}(\varphi)\tilde{U}_y^* h](x) =&\pi(\omega(x,y))[\tilde{\pi}(\varphi)\tilde{U}_y^* h](x+y)\\ =&\pi(\omega(x,y))\pi(\theta_{x+y}(\varphi))[\tilde{U}_y^* h](x+y)\\ =&\pi(\omega(x,y))\pi(\theta_{x+y}(\varphi))\pi(\omega(x+y-y,y)^*)h(x+y-y)\\ =&\pi(\omega(x,y)\theta_{x+y}(\varphi)\omega(x,y)^*)h(x)\\ =&\pi(\theta_{x+y}(\varphi))h(x) = \pi(\theta_x(\theta_y(\varphi)))h(x) = [\tilde{\pi}(\theta_y(\varphi))h](x).\end{aligned}$$

Exercise 6.1.1 *Check that if $f(x,\xi) = f(\xi)$ (f is independent of x), then $\mathfrak{Op}(f) = f(D)$, while if $f(x,\xi) = f(x)$ (f is independent of ξ), then $\mathfrak{Op}(f) = f(X)$.*

Let f be independent of x. Then we get for any $u \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
[\mathfrak{Dp}(f)u](x) =& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}, \eta\right) u(y)\, dy d\eta \\
=& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f(\eta) u(y)\, dy d\eta \\
=& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{iz\cdot\eta} f(\eta) u(x-z)\, dz d\eta \\
=& \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \int_{\hat{\mathbb{R}}^d} e^{iz\cdot\eta} f(\eta)\, d\eta u(x-z)\, dz \\
=& \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \check{f}(z) u(x-z)\, dz \\
=& [\check{f} * u](x) = [f(D)u](x).
\end{aligned}$$

On the other hand, let f be independent of ξ. Then we get for any $u \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
[\mathfrak{Dp}(f)u](x) =& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}, \eta\right) u(y)\, dy d\eta \\
=& \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}\right) u(y)\, dy d\eta \\
=& \int_{\mathbb{R}^d} \delta(y-x) f\left(\frac{x+y}{2}\right) u(y)\, dy \\
=& f(x)u(x) = [f(X)u](x).
\end{aligned}$$

Exercise 7.2.1 *Check the above relation.*

Let $\mathsf{x} = (x, \xi)$, $\mathsf{y} = (y, \eta) \in \Xi$. Then for any $u \in \mathcal{H}$ and $z \in \mathbb{R}^n$, we get

$$\begin{aligned}
[W^A(\mathsf{x})W^A(\mathsf{y})u](z) =& e^{-\frac{i}{2}x\cdot\xi}[V_\xi U^A(x)W^A(\mathsf{y})u](z) \\
=& e^{-\frac{i}{2}x\cdot\xi}e^{-iz\cdot\xi}[U^A(x)W^A(\mathsf{y})u](z) \\
=& e^{-\frac{i}{2}x\cdot\xi}e^{-iz\cdot\xi}\lambda^A(z;x)[W^A(\mathsf{y})u](x+z) \\
=& e^{-\frac{i}{2}x\cdot\xi}e^{-iz\cdot\xi}\lambda^A(z;x)e^{-\frac{i}{2}y\cdot\eta}[V_\eta U^A(y)u](x+z) \\
=& e^{-\frac{i}{2}x\cdot\xi}e^{-iz\cdot\xi}\lambda^A(z;x)e^{-\frac{i}{2}y\cdot\eta}e^{i(x+z)\cdot\eta}[U^A(y)u](x+z) \\
=& e^{-\frac{i}{2}x\cdot\xi}e^{-iz\cdot\xi}\lambda^A(z;x)e^{-\frac{i}{2}y\cdot\eta}e^{-i(x+z)\cdot\eta}\lambda^A(x+z;y)u(x+y+z) \\
=& \exp(-\frac{i}{2}x\cdot\xi - \frac{i}{2}y\cdot\eta - ix\cdot\eta - iz\cdot\xi - iz\cdot\eta) \\
& \quad \lambda^A(z;x)\theta_x(\lambda^A(z;y))u(x+y+z).
\end{aligned}$$

Since we have $\delta^1(\lambda^A(z))(x,y)\omega^B(z;x,y)$, we get

$$\lambda^A(z;x)\theta_x(\lambda^A(z;y)) = \omega^B(z;x,y)\lambda^A(z;x+y).$$

Besides, we have

$$\exp(-\frac{i}{2}x\cdot\xi - \frac{i}{2}y\cdot\eta - ix\cdot\eta - iz\cdot\xi - iz\cdot\eta) = \exp(\frac{i}{2}\sigma(\mathsf{x},\mathsf{y}) - \frac{i}{2}(x+y)\cdot(\xi+\eta) - iz\cdot(\xi+\eta))$$

Hence we get

$$\begin{aligned}
[W^A(\mathsf{x})W^A(\mathsf{y})u](z) =& \exp(-\frac{i}{2}x\cdot\xi - \frac{i}{2}y\cdot\eta - ix\cdot\eta - iz\cdot\xi - iz\cdot\eta) \\
& \quad \lambda^A(z;x)\theta_x(\lambda^A(z;y))u(x+y+z) \\
=& e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}e^{-\frac{i}{2}(x+y)\cdot(\xi+\eta)}e^{-iz\cdot(\xi+\eta)} \\
& \quad \omega^B(z;x,y)\lambda^A(z;x+y)u(x+y+z) \\
=& e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\omega^B(z;x,y)e^{-\frac{i}{2}(x+y)\cdot(\xi+\eta)}e^{-iz\cdot(\xi+\eta)}[U^A(x+y)u](z) \\
=& e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\omega^B(z;x,y)e^{-\frac{i}{2}(x+y)\cdot(\xi+\eta)}[V_{\xi+\eta}U^A(x+y)u](z) \\
=& e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\omega^B(z;x,y)[W^A(\mathsf{x}+\mathsf{y})u](z) \\
=& e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}[\pi(\omega^B(x,y))W^A(\mathsf{x}+\mathsf{y})u](z)
\end{aligned}$$

since π is defined by $\pi(a) = a(X)$.