C^*-algebraic methods in spectral theory

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Contents

Chapter 1

Linear operators on a Hilbert space

This chapter is mainly based on the first chapters of the book [Amr09]. All missing proofs can be found in this reference.

1.1 Hilbert space

Definition 1.1.1. A (complex) Hilbert space $\mathcal{H}$ *is a vector space on* $\mathbb{C}$ *with a strictly positive scalar product (or inner product), which is complete for the associated norm and which admits a countable basis. The scalar product is denoted by* $\langle\cdot,\cdot\rangle$ *and the corresponding norm by* $\|\cdot\|$.

In particular, note that for any $f, g, h \in \mathcal{H}$ and $\alpha \in \mathbb{C}$ the following properties hold:

(i) $\langle f, g\rangle = \overline{\langle g, f\rangle}$,

(ii) $\langle f + \alpha g, h\rangle = \langle f, h\rangle + \alpha\langle g, h\rangle$,

(iii) $\|f\|^2 = \langle f, f\rangle > 0$ if and only if $f \neq 0$.

From now on, the symbol $\mathcal{H}$ will always denote a Hilbert space.

Examples 1.1.2. *(i)* $\mathcal{H} = \mathbb{C}^d$ *with* $\langle\alpha, \beta\rangle = \sum_{j=1}^{d} \alpha_j \overline{\beta_j}$ *for any* $\alpha, \beta \in \mathbb{C}^d$,

(ii) $\mathcal{H} = l^2(\mathbb{Z})$ *with* $\langle a, b\rangle = \sum_{j\in\mathbb{Z}} a_j \overline{b_j}$ *for any* $a, b \in l^2(\mathbb{Z})$,

(iii) $\mathcal{H} = L^2(\mathbb{R}^d)$ *with* $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$ *for any* $f, g \in L^2(\mathbb{R}^d)$.

Let us recall some useful inequalities: For any $f, g \in \mathcal{H}$ one has

(i) $|\langle f, g\rangle| \leq \|f\|\,\|g\|$ Schwarz inequality,

(ii) $\|f + g\| \leq \|f\| + \|g\|$,

(iii) $\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$,

(iv) $\big| \|f\| - \|g\| \big| \leq \|f - g\|$

the last 3 inequalities are called triangle inequalities. In addition, let us recall that $f, g \in \mathcal{H}$ are said *orthogonal* if $\langle f, g \rangle = 0$.

Definition 1.1.3. *A sequence* $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ *is* strongly convergent *to* $f_\infty \in \mathcal{H}$ *if* $\lim_{n\to\infty} \|f_n - f_\infty\| = 0$*, or is* weakly convergent *to* $f_\infty \in \mathcal{H}$ *if for any* $g \in \mathcal{H}$ *one has* $\lim_{n\to\infty} \langle f_n - f_\infty, g \rangle = 0$*.*

Clearly, a strongly convergent sequence is also weakly convergent. The converse is not true.

Definition 1.1.4. A subspace $\mathcal{M}$ *of a Hilbert space* $\mathcal{H}$ *is a linear subset of* $\mathcal{H}$*, or more precisely* $\forall f, g \in \mathcal{M}$ *and* $\alpha \in \mathbb{C}$ *one has* $f + \alpha g \in \mathcal{M}$*.*

Note that if $\mathcal{M}$ is closed, then $\mathcal{M}$ is a Hilbert space in itself, with the scalar product and norm inherited from $\mathcal{H}$.

Examples 1.1.5. *(i) If* $f_1, \ldots, f_n \in \mathcal{H}$*, then* $\mathsf{Vect}(f_1, \ldots, f_n)$ *is the closed vector space generated by the linear combinations of* $f_1, \ldots f_n$*.* $\mathsf{Vect}(f_1, \ldots, f_n)$ *is a closed subspace.*

(ii) If $\mathcal{M}$ *is a closed subspace of* $\mathcal{H}$*, then* $\mathcal{M}^\perp := \{f \in \mathcal{H} \mid \langle f, g \rangle = 0, \forall g \in \mathcal{M}\}$ *is a closed subspace of* $\mathcal{H}$*.*

Note that the closed subspace $\mathcal{M}^\perp$ is called *the orthocomplement of* $\mathcal{M}$ *in* $\mathcal{H}$. Indeed, one has:

Lemma 1.1.6 (Projection Theorem)**.** *Let* $\mathcal{M}$ *be a closed subspace of a Hilbert space* $\mathcal{H}$*. Then, for any* $f \in \mathcal{H}$ *there exist a unique* $f_1 \in \mathcal{M}$ *and a unique* $f_2 \in \mathcal{M}^\perp$ *such that* $f = f_1 + f_2$*.*

Let us recall that the dual $\mathcal{H}^*$ of the Hilbert space $\mathcal{H}$ consists in the set of all bounded linear functionals on $\mathcal{H}$, *i.e.* $\mathcal{H}^*$ consists in all mappings $\varphi : \mathcal{H} \to \mathbb{C}$ satisfying for any $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$

(i) $\varphi(f + \alpha g) = \varphi(f) + \alpha\varphi(g)$, (linearity)

(ii) $|\varphi(f)| \leq c\|f\|$, (boundedness)

where c is a constant independent of f. One sets

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{0 \neq f \in \mathcal{H}} \frac{|\varphi(f)|}{\|f\|}.$$

Note that if $g \in \mathcal{H}$, then g defines an element φ_g of $\mathcal{H}^*$ by setting $\varphi_g(f) := \langle f, g \rangle$.

Lemma 1.1.7 (Riesz Lemma)**.** *For any* $\varphi \in \mathcal{H}^*$*, there exists a unique* $g \in \mathcal{H}$ *such that for any* $f \in \mathcal{H}$

$$\varphi(f) = \langle f, g \rangle.$$

In addition, g *satisfies* $\|\varphi\|_{\mathcal{H}^*} = \|g\|$*.*

As a consequence, one often identifies $\mathcal{H}^*$ with $\mathcal{H}$ itself.

1.2 Bounded operators

First of all, let us recall that a linear map B between two complex vector spaces $\mathcal{M}$ and $\mathcal{N}$ satisfies $B(f+\alpha g) = Bf + \alpha Bg$ for all $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{C}$.

Definition 1.2.1. *A map $B : \mathcal{H} \to \mathcal{H}$ is* a bounded linear operator *if $B : \mathcal{H} \to \mathcal{H}$ is a linear map, and if there exists $c \in \mathbb{R}$ such that $\|Bf\| \leq c\|f\|$ for all $f \in \mathcal{H}$. The set of* all bounded linear operators on $\mathcal{H}$ *is denoted by $\mathscr{B}(\mathcal{H})$.*

For any $B \in \mathscr{B}(\mathcal{H})$, one sets

$$\|B\| := \sup_{0 \neq f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}. \tag{1.2.1}$$

and call it *the norm of B*. Note that the same notation is used for the norm of an element of $\mathcal{H}$ and for the norm of an element of $\mathscr{B}(\mathcal{H})$, but this does not lead to any confusion.

Lemma 1.2.2. *If $B \in \mathscr{B}(\mathcal{H})$, then $\|B\| = \sup_{f,g \in \mathcal{H} \text{ with } \|f\|=\|g\|=1} |\langle Bf, g\rangle|$.*

Definition 1.2.3. *A sequence $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathcal{H})$ is* uniformly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$, is* strongly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f \in \mathcal{H}$ one has $\lim_{n\to\infty} \|B_n f - B_\infty f\| = 0$, or is* weakly convergent *to $B_\infty \in \mathscr{B}(\mathcal{H})$ if for any $f, g \in \mathcal{H}$ one has $\lim_{n\to\infty}\langle B_n f - B_\infty f, g\rangle = 0$. In these cases, one writes respectively $u-\lim_{n\to\infty} B_n = B_\infty$, $s-\lim_{n\to\infty} B_n = B_\infty$ and $w-\lim_{n\to\infty} B_n = B_\infty$.*

Clearly, uniform convergence implies strong convergence, and strong convergence implies weak convergence. The reverse statements are not true.

Lemma 1.2.4. *For any $B \in \mathscr{B}(\mathcal{H})$, there exists a unique $B^* \in \mathscr{B}(\mathcal{H})$ such that for any $f, g \in \mathcal{H}$*

$$\langle Bf, g\rangle = \langle f, B^* g\rangle.$$

The operator B^* is called *the adjoint of B*, and the proof of this statement involves the Riesz Lemma.

Proposition 1.2.5. *The following properties hold:*

(i) *$\mathscr{B}(\mathcal{H})$ is an algebra,*

(ii) *The map $\mathscr{B}(\mathcal{H}) \ni B \mapsto B^* \in \mathscr{B}(\mathcal{H})$ is an involution,*

(iii) *$\mathscr{B}(\mathcal{H})$ is complete with the norm $\|\cdot\|$,*

(iv) *One has $\|B^*\| = \|B\|$ and $\|B^*B\| = \|B\|^2$.*

As a consequence of these properties, $\mathscr{B}(\mathcal{H})$ is a C^*-algebra, as we shall see later on.

Definition 1.2.6. *For any $B \in \mathscr{B}(\mathcal{H})$ one sets*

$$\mathsf{Ran}(B) := B\mathcal{H} = \{f \in \mathcal{H} \mid f = Bg \text{ for some } g \in \mathcal{H}\},$$

and call this set the range of B.

Definition 1.2.7. *An operator $B \in \mathscr{B}(\mathcal{H})$ is* invertible *if the equation $Bf = 0$ only admits the solution $f = 0$. In such a case, there exists a linear map $B^{-1} : \mathsf{Ran}(B) \to \mathcal{H}$ which satisfies $B^{-1}Bf = f$ for any $f \in \mathcal{H}$, and $BB^{-1}g = g$ for any $g \in \mathsf{Ran}(B)$. If B is invertible and $\mathsf{Ran}(B) = \mathcal{H}$, then $B^{-1} \in \mathscr{B}(\mathcal{H})$ and B is said* boundedly invertible *or* invertible in $\mathscr{B}(\mathcal{H})$.

Note that the two conditions B invertible and $\mathsf{Ran}(B) = \mathcal{H}$ imply $B^{-1} \in \mathscr{B}(\mathcal{H})$ is a consequence of the Closed graph Theorem.

Remark 1.2.8. *In the sequel, we shall use the notation $\mathbf{1} \in \mathscr{B}(\mathcal{H})$ for the operator defined on any $f \in \mathcal{H}$ by $\mathbf{1}f = f$, and $\mathbf{0} \in \mathscr{B}(\mathcal{H})$ for the operator defined by $\mathbf{0}f = 0$.*

Lemma 1.2.9 (Neumann series)**.** *If $B \in \mathscr{B}(\mathcal{H})$ and $\|B\| < 1$, then the operator $(\mathbf{1}-B)$ is invertible in $\mathscr{B}(\mathcal{H})$, with*

$$(\mathbf{1} - B)^{-1} = \sum_{n=0}^{\infty} B^n,$$

and with $\big\|(\mathbf{1} - B)^{-1}\big\| \leq (1 - \|B\|)^{-1}$.

Note that we have used the identity $B^0 = \mathbf{1}$.

1.3 Special classes of operators

Definition 1.3.1. *An element $U \in \mathscr{B}(\mathcal{H})$ is* a unitary operator *if $UU^* = \mathbf{1}$ and if $U^*U = \mathbf{1}$.*

Note that in this case, U is boundedly invertible with $U^{-1} = U^*$. Indeed, observe first that $Uf = 0$ implies $f = U^*(Uf) = U^*0 = 0$. Secondly, for any $g \in \mathcal{H}$, one has $g = U(U^*g)$, and thus $\mathsf{Ran}(U) = \mathcal{H}$. Finally, the equality $U^{-1} = U^*$ follows from the unicity of the inverse.

Definition 1.3.2. *An element $P \in \mathscr{B}(\mathcal{H})$ is* an orthogonal projection *if $P = P^2 = P^*$.*

In this case, $P\mathcal{H}$ is a closed subspace of $\mathcal{H}$. Alternatively, for each closed subspace $\mathcal{M}$ of $\mathcal{H}$, there exists an orthogonal projection P such that $P\mathcal{H} = \mathcal{M}$.

Now, for any family $\{g_j, h_j\}_{j=1}^n \subset \mathcal{H}$ and for any $f \in \mathcal{H}$ one sets

$$A_n f := \sum_{j=1}^{n} \langle f, g_j\rangle\, h_j. \tag{1.3.1}$$

Then $A_n \in \mathscr{B}(\mathcal{H})$, and $\mathsf{Ran}(A_n) \subset \mathsf{Vect}(h_1, \ldots, h_n)$. Such an operator A_n is called *a finite rank operator*. In fact, any operator $B \in \mathscr{B}(\mathcal{H})$ with $\dim\big(\mathsf{Ran}(B)\big) < \infty$ is a finite rank operator.

Exercise 1.3.3. *For the operator A_n defined in* (1.3.1)*, give an upper estimate for $\|A_n\|$ and compute A_n^*.*

Definition 1.3.4. *An element $B \in \mathscr{B}(\mathcal{H})$ is* a compact operator *if there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of finite rank operators such that $\lim_{n\to\infty} \|A_n - B\| = 0$. The set of all compact operators is denoted by $\mathscr{K}(\mathcal{H})$.*

Proposition 1.3.5. *The following properties hold:*

(i) $B \in \mathscr{K}(\mathcal{H}) \Longleftrightarrow B^* \in \mathscr{K}(\mathcal{H})$,

(ii) $\mathscr{K}(\mathcal{H})$ *is a $*$-algebra, complete for the norm* $\|\cdot\|$,

(iii) If $B \in \mathscr{K}(\mathcal{H})$ and $A \in \mathscr{B}(\mathcal{H})$, then AB and BA belong to $\mathscr{K}(\mathcal{H})$.

As a consequence, $\mathscr{K}(\mathcal{H})$ is a C^*-algebra and an ideal of $\mathscr{B}(\mathcal{H})$.

Extension 1.3.6. *There are various subalgebras of $\mathscr{K}(\mathcal{H})$, for example the algebra of Hilbert-Schmidt operators, the algebra of trace class operators, and more generally the Schatten classes. Note that these algebras are not closed with respect to the norm $\|\cdot\|$ but with respect to some stronger norms $|||\cdot|||$. These algebras are ideals in $\mathscr{B}(\mathcal{H})$.*

1.4 Operator valued maps

Let I be an open interval on $\mathbb{R}$, and let us consider a map $F : I \to \mathscr{B}(\mathcal{H})$.

Definition 1.4.1. *The map F is* continuous in norm *on I if for all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\|F(x+\varepsilon) - F(x)\big\| = 0.$$

The map F is strongly continuous *on I if for any $f \in \mathcal{H}$ and all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\|F(x+\varepsilon)f - F(x)f\big\| = 0.$$

The map F is weakly continuous *on I if for any $f, g \in \mathcal{H}$ and all $x \in I$*

$$\lim_{\varepsilon\to 0} \big\langle \big(F(x+\varepsilon) - F(x)\big)f, g\big\rangle = 0.$$

One writes respectively $u - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$, $s - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$ and $w - \lim_{\varepsilon\to 0} F(x+\varepsilon) = F(x)$.

Definition 1.4.2. *The map F is* differentiable in norm *on I if there exists a map $F' : I \to \mathscr{B}(\mathcal{H})$ such that*

$$\lim_{\varepsilon\to 0} \Big\|\frac{1}{\varepsilon}\big(F(x+\varepsilon) - F(x)\big) - F'(x)\Big\| = 0.$$

The definitions for strongly differentiable *and* weakly differentiable *are similar.*

If I is an open interval of $\mathbb{R}$ and if $F : I \to \mathscr{B}(\mathcal{H})$, one defines $\int_I F(x)\,\mathrm{d}x$ as a Riemann integral (limit of finite sums over a partition of I) if this limiting procedure exists and is independent of the partitions of I. Note that these integrals can be defined in the weak topology, in the strong topology or in the norm topology (and in other topologies). For example, if $F : I \to \mathscr{B}(\mathcal{H})$ is strongly continuous and if $\int_I \|F(x)\|\mathrm{d}x < \infty$, then the integral $\int_I F(x)\,\mathrm{d}x$ exists in the strong topology.

Proposition 1.4.3. *Let I is an open interval of $\mathbb{R}$ and $F : I \to \mathscr{B}(\mathcal{H})$ such that $\int_I F(x)\,\mathrm{d}x$ exists (in an appropriate topology). Then,*

(i) For any $B \in \mathscr{B}(\mathcal{H})$ one has

$$B\int_I F(x)\,\mathrm{d}x = \int_I BF(x)\,\mathrm{d}x \quad \text{and} \quad \Big(\int_I F(x)\,\mathrm{d}x\Big)B = \int_I F(x)B\,\mathrm{d}x,$$

(ii) one also has $\Big\|\int_I F(x)\,\mathrm{d}x\Big\| \leq \int_I \|F(x)\|\,\mathrm{d}x$,

(iii) If $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ is a subalgebra of $\mathscr{B}(\mathcal{H})$, closed with respect to a norm $|||\cdot|||$, and if the map $F : I \to \mathscr{C}$ is continuous with respect to this norm and satisfies $\int_I |||F(x)|||\,\mathrm{d}x < \infty$, then $\int_I F(x)\,\mathrm{d}x$ exists, belongs to $\mathscr{C}$ and satisfies

$$\Big|\Big|\Big|\int_I F(x)\,\mathrm{d}x\Big|\Big|\Big| \leq \int_I |||F(x)|||\,\mathrm{d}x.$$

Note that the last statement is very useful, for example when $\mathscr{C} = \mathscr{K}(\mathcal{H})$ or any Schatten class.

1.5 Unbounded operators

In this section, we define an extension of the notion of bounded linear operators. Obviously, the following definitions and results are also valid for bounded linear operators.

Definition 1.5.1. *A* linear operator on *$\mathcal{H}$ is a pair $\big(A, \mathsf{D}(A)\big)$, where $\mathsf{D}(A)$ is a subspace of $\mathcal{H}$ and A is a linear map from $\mathsf{D}(A)$ to $\mathcal{H}$. $\mathsf{D}(A)$ is called* the domain of A. *One says that the operator $\big(A, \mathsf{D}(A)\big)$ is* densely defined *if $\mathsf{D}(A)$ is dense in $\mathcal{H}$.*

Note that one often just says *the linear operator* A, but that its domain $\mathsf{D}(A)$ is implicitly taken into account. For such an operator, its range $\mathsf{Ran}(A)$ is defined by

$$\mathsf{Ran}(A) := A\,\mathsf{D}(A) = \{f \in \mathcal{H} \mid f = Ag \text{ for some } g \in \mathsf{D}(A)\}.$$

In addition, one defines the kernel $\mathsf{Ker}(A)$ of A by

$$\mathsf{Ker}(A) := \{f \in \mathsf{D}(A) \mid Af = 0\}.$$

Example 1.5.2. *Let $\mathcal{H} := L^2(\mathbb{R})$ and consider the operator X defined by $[Xf](x) = xf(x)$ for any $x \in \mathbb{R}$. Clearly, $\mathsf{D}(X) = \{f \in \mathcal{H} \mid \int_{\mathbb{R}} |xf(x)|^2 \,\mathrm{d}x < \infty\} \subsetneq \mathcal{H}$. In addition, by considering the family of functions $\{f_y\}_{y\in\mathbb{R}} \subset \mathsf{D}(X)$ with $f_y(x) := e^{|x-y|^2}$, one easily observes that* $\sup_{0\neq f\in\mathsf{D}(X)} \frac{\|Xf\|}{\|f\|} = \infty$, *which can be compared with* (1.2.1).

Definition 1.5.3. *For any pair of linear operators $\big(A, \mathsf{D}(A)\big)$ and $\big(B, \mathsf{D}(B)\big)$ satisfying $\mathsf{D}(A) \subset \mathsf{D}(B)$ and $Af = Bf$ for all $f \in \mathsf{D}(A)$, one says that $\big(B, \mathsf{D}(B)\big)$ is* an extension of $\big(A, \mathsf{D}(A)\big)$ *to $\mathsf{D}(B)$, or that $\big(A, \mathsf{D}(A)\big)$ is the restriction of $\big(B, \mathsf{D}(B)\big)$ to $\mathsf{D}(A)$.*

Let us note that if $\big(A, \mathsf{D}(A)\big)$ is densely defined and if there exists $c \in \mathbb{R}$ such that $\|Af\| \leq c\|f\|$ for all $f \in \mathsf{D}(A)$, then there exists a natural continuous extension $\overline{A}$ of A with $\mathsf{D}(\overline{A}) = \mathcal{H}$. This extension satisfies $\overline{A} \in \mathscr{B}(\mathcal{H})$ with $\|\overline{A}\| \leq c$, and is called *the closure of the operator A*.

Exercise 1.5.4. *Construct this natural extension and show that $\|\overline{A}\| \leq c$.*

Let us stress that the sum $A + B$ for two linear operators is *a priori* only defined on the subspace $\mathsf{D}(A) \cap \mathsf{D}(B)$, and that the product AB is *a priori* defined only on the subspace $\{f \in \mathsf{D}(B) \mid Bf \in \mathsf{D}(A)\}$. These two sets can be very small.

Definition 1.5.5. *Let $\big(A, \mathsf{D}(A)\big)$ be a densely defined linear operator on $\mathcal{H}$.* The adjoint A^* of A *is the operator defined by*

$$\mathsf{D}(A^*) := \big\{f \in \mathcal{H} \mid \exists f^* \in \mathcal{H} \text{ with } \langle f^*, g\rangle = \langle f, Ag\rangle \text{ for all } g \in \mathsf{D}(A)\big\}$$

and $A^*f := f^*$ for all $f \in \mathsf{D}(A^*)$.

Let us note that the density of $\mathsf{D}(A)$ is necessary to ensure that A^* is well defined. Indeed, if f_1^*, f_2^* satisfy for all $g \in \mathsf{D}(A)$

$$\langle f_1^*, g\rangle = \langle f, Ag\rangle = \langle f_2^*, g\rangle,$$

then $\langle f_1^* - f_2^*, g\rangle = 0$ for all $g \in \mathsf{D}(A)$, and this equality implies $f_1^* = f_2^*$ only if $\mathsf{D}(A)$ is dense in $\mathcal{H}$. Note also that once $\big(A^*, \mathsf{D}(A^*)\big)$ is defined, one has

$$\langle A^*f, g\rangle = \langle f, Ag\rangle \qquad \forall f \in \mathsf{D}(A^*) \text{ and } \forall g \in \mathsf{D}(A).$$

Lemma 1.5.6. *Let $\big(A, \mathsf{D}(A)\big)$ be a densely defined linear operator on $\mathcal{H}$. Then*

$$\mathsf{Ker}(A^*) = \mathsf{Ran}(A)^\perp.$$

Proof. Let $f \in \mathsf{Ker}(A^*)$, *i.e.* $f \in \mathsf{D}(A^*)$ and $A^*f = 0$. Then, for all $g \in \mathsf{D}(A)$, one has

$$0 = \langle A^*f, g\rangle = \langle f, Ag\rangle$$

meaning that $f \in \mathsf{Ran}(A)^\perp$. Conversely, if $f \in \mathsf{Ran}(A)^\perp$, then for all $g \in \mathsf{D}(A)$ one has

$$\langle f, Ag\rangle = 0 = \langle 0, g\rangle$$

meaning that $f \in \mathsf{D}(A^*)$ and $A^*f = 0$, by the definition of the adjoint of A. □

Definition 1.5.7. *A densely defined linear operator* $\big(A, \mathsf{D}(A)\big)$ *is* self-adjoint *if* $\mathsf{D}(A^*) = \mathsf{D}(A)$ *and* $A^* f = Af$ *for all* $f \in \mathsf{D}(A)$.

Note that whenever the operator A is self-adjoint one has

$$\langle Af, g\rangle = \langle f, Ag\rangle \qquad \forall f, g \in \mathsf{D}(A).$$

Let us stress that self-adjoint operators are very important in relation with quantum mechanics: any physical system is described with such an operator. Self-adjoint operators are the natural generalisation of Hermitian matrices.

Extension 1.5.8. *Self-adjoint operators are a special class of closed and symmetric linear operators. These notions, as well as the graph or the essential self-adjointness of an operator are important topics for the study of unbounded linear operators.*

1.6 Resolvent and spectrum

Definition 1.6.1. *For a closed*[1] *linear operator* A*, a value* $z \in \mathbb{C}$ *is* an eigenvalue of A *if there exists* $f \in \mathsf{D}(A)$*,* $f \neq 0$*, such that* $Af = zf$*. In such a case, the element* f *is called* an eigenfunction of A associated with the eigenvalue z*. The set of all eigenvalues of* A *is denoted by* $\sigma_p(A)$.

Lemma 1.6.2. *Let* A *be a self-adjoint operator on* $\mathcal{H}$*. Then,*

(i) All eigenvalues of A *are real,*

(ii) Two eigenfunctions of A *associated with two different eigenvalues of* A *are orthogonal.*

Proof. (i) Assume that $Af = zf$ for some $z \in \mathbb{C}$ and $f \in \mathsf{D}(A)$ with $f \neq 0$. Then, one has

$$z\|f\|^2 = \langle zf, f\rangle = \langle Af, f\rangle = \langle f, Af\rangle = \langle f, zf\rangle = \overline{z}\|f\|^2,$$

which implies that $z \in \mathbb{R}$.

(ii) Assume that $Af = \lambda f$ and that $Ag = \mu g$ with $\lambda, \mu \in \mathbb{R}$ and $\lambda \neq \mu$, and $f, g \in \mathsf{D}(A)$, with $f \neq 0$ and $g \neq 0$. Then

$$\lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, Ag\rangle = \mu\langle f, g\rangle,$$

which implies that $\langle f, g\rangle = 0$, or in other words that f and g are orthogonal. □

[1] An operator A is closed if the three conditions (i) $f_n \in \mathsf{D}(A)$, (ii) $s-\lim_{n\to\infty} f_n = f$, (iii) $\{Af_n\}$ is strongly Cauchy, imply that $f \in \mathsf{D}(A)$ and $s-\lim_{n\to\infty} Af_n = Af$. Note that any self-adjoint operator as well as any bounded operator is closed.

By analogy to the bounded case, we say that A is *invertible* if $\mathsf{Ker}(A) = \{0\}$. In this case, the inverse A^{-1} gives a bijection from $\mathsf{Ran}(A)$ onto $\mathsf{D}(A)$. Note now that if z is an eigenvalue of a linear operator A, then $(A - z)$ is not invertible since $(A - z)f = 0$ for some $f \in \mathsf{D}(A)$ with $f \neq 0$. Then, *the spectrum of the operator* A is a generalization of the notion of eigenvalues which is based on the previous observation.

Definition 1.6.3. The resolvent set $\rho(A)$ *of a closed linear operator* A *is defined by*

$$\begin{aligned}\rho(A) :=& \big\{z \in \mathbb{C} \mid (A - z) \textit{ is invertible in } \mathscr{B}(\mathcal{H})\big\}\\ =& \big\{z \in \mathbb{C} \mid \mathsf{Ker}(A - z) = \{0\} \textit{ and } \mathsf{Ran}(A - z) = \mathcal{H}\big\}.\end{aligned}$$

The spectrum $\sigma(A)$ of A *is the complement of* $\rho(A)$ *in* $\mathbb{C}$, i.e. $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Definition 1.6.4. *For any closed linear operator* A *and for any* $z \in \rho(A)$*, the operator* $(A - z)^{-1} \in \mathscr{B}(\mathcal{H})$ *is called* the resolvent of A at the point z.

Exercise 1.6.5. *For any closed linear operator* A *and any* $z_1, z_2 \in \rho(A)$*, show the* first resolvent equation*, namely*

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}. \tag{1.6.1}$$

Lemma 1.6.6. *The spectrum of a self-adjoint operator* A *is real,* i.e. $\sigma(A) \subset \mathbb{R}$.

Proof of Lemma 1.6.6. Let us consider $z = \lambda + i\varepsilon$ with $\varepsilon \neq 0$, and show that $z \in \rho(A)$. Indeed, for any $f \in \mathsf{D}(A)$ one has

$$\begin{aligned}\|(A - z)f\|^2 &= \|(A - \lambda)f - i\varepsilon f\|^2\\ &= \big\langle (A - \lambda)f - i\varepsilon f, (A - \lambda)f - i\varepsilon f\big\rangle\\ &= \|(A - \lambda)f\|^2 + \varepsilon^2\|f\|^2.\end{aligned}$$

It follows that $\|(A - z)f\| \geq |\varepsilon|\|f\|$, and thus $A - z$ is invertible.

Now, for any for any $g \in \mathsf{Ran}(A - z)$ let us observe that

$$\|g\| = \big\|(A - z)(A - z)^{-1}g\big\| \geq |\varepsilon|\big\|(A - z)^{-1}g\big\|.$$

Equivalently, it means for all $g \in \mathsf{Ran}(A - z)$, one has

$$\big\|(A - z)^{-1}g\big\| \leq \frac{1}{|\varepsilon|}\|g\|. \tag{1.6.2}$$

Let us finally observe that $\mathsf{Ran}(A - z)$ is dense in $\mathcal{H}$. Indeed, by Lemma 1.5.6 one has

$$\mathsf{Ran}(A - z)^{\perp} = \mathsf{Ker}\,\big((A - z)^*\big) = \mathsf{Ker}(A^* - \overline{z}) = \mathsf{Ker}(A - \overline{z}) = \{0\}$$

since all eigenvalues of A are real. Thus, the operator $(A - z)^{-1}$ is defined on the dense domain $\mathsf{Ran}(A - z)$ and satisfies the estimate (1.6.2). As explained just before the Exercise 1.5.4, it means that $(A - z)^{-1}$ continuously extends to an element of $\mathscr{B}(\mathcal{H})$, and therefore $z \in \rho(A)$. □

1.7 Spectral theory for self-adjoint operators

1.7.1 Stieltjes measures

Let us consider a function $F : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

(i) F is monotone non-decreasing, *i.e.* $\lambda \geq \mu \Longrightarrow F(\lambda) \geq F(\mu)$,

(ii) F is right continuous, *i.e.* $F(\lambda) = F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ for all $\lambda \in \mathbb{R}$,

(iii) $F(-\infty) := \lim_{\lambda \to -\infty} F(\lambda) = 0$ and $\rho := F(+\infty) := \lim_{\lambda \to \infty} F(\lambda) < \infty$.

Note that $F(\lambda + 0) := \lim_{\varepsilon \searrow 0} F(\lambda + \varepsilon)$ and $F(\lambda - 0) := \lim_{\varepsilon \searrow 0} F(\lambda - \varepsilon)$ exist since F is a monotone and bounded function.

With a function F having these properties, one can associate a bounded Borel measure m_F on $\mathbb{R}$, called *Stieltjes measure*, starting with

$$m_F\big((a,b]\big) := F(b) - F(a), \qquad a, b \in \mathbb{R}$$

and extending then this definition to all Borel sets of $\mathbb{R}$. With this definition, note that $m_F(\mathbb{R}) = \rho$ and that

$$m_F\big((a,b)\big) = F(b-0) - F(a), \qquad m_F\big([a,b]\big) = F(b) - F(a-0)$$

and therefore $m_F\big(\{a\}\big) = F(a) - F(a-0)$ is different from 0 if F is not continuous at the point a.

Note that starting with a bounded Borel measure m on $\mathbb{R}$ and setting $F(\lambda) := m\big((-\infty, \lambda]\big)$, then F satisfies the conditions (i)-(iii) and the associated Stieltjes measure m_F verifies $m_F = m$.

Theorem 1.7.1. *Any Stieltjes measure m admits a unique decomposition*

$$m = m_p + m_{ac} + m_{sc}$$

where m_p is a pure point measure, *m_{ac} is* an absolutely continuous measure *with respect to the Lebesgue measure on $\mathbb{R}$, and m_{sc} is* a singular continuous measure *with respect to the Lebesgue measure $\mathbb{R}$.*

This result is based on *Lebesgue Decomposition Theorem*. Let us simply stress that m_{sc} is singular with respect to the Lebesgue measure but $m_{sc}(\{\lambda\}) = 0$ for any $\lambda \in \mathbb{R}$. On the other hand, for any Borel set V, $m_p(V) = \sum_{\lambda \in V} m(\{\lambda\})$, where this sum contains at most a countable number of contributions.

1.7.2 Spectral measures

We shall now define a spectral measure, by analogy with the Stieltjes measure defined in the previous section. For an arbitrary Hilbert space $\mathcal{H}$ we shall write $\mathcal{P}(\mathcal{H})$ for the set of orthogonal projections in $\mathcal{H}$.

Definition 1.7.2. A spectral family, *or* a resolution of the identity, *is a family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *of orthogonal projections in* $\mathcal{H}$ *satisfying:*

(i) The family is non-decreasing, i.e. $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$,

(ii) The family is strongly right continuous, i.e. $E_\lambda = E_{\lambda+0} = s-\lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$,

(iii) $s-\lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ *and* $s-\lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$,

It is important to observe that the condition (i) implies that the elements of the families are *commuting, i.e.* $E_\lambda E_\mu = E_\mu E_\lambda$. We also define *the support of the spectral family* as the following subset of $\mathbb{R}$:

$$\operatorname{supp}\{E_\lambda\} = \{\mu\in\mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq \mathbf{0},\ \forall\varepsilon>0\}.$$

With such a spectral family one first defines

$$E\big((a,b]\big) := E_b - E_a, \qquad a,b\in\mathbb{R}, \tag{1.7.1}$$

and extends this definition to all Borel sets on $\mathbb{R}$ (we denote by $\mathcal{A}_B$ the set of all Borel sets on $\mathbb{R}$). One ends up with a projection-valued map $E:\mathcal{A}_B\to\mathcal{P}(\mathcal{H})$ which satisfies $E(\emptyset)=\mathbf{0}$, $E(\mathbb{R})=\mathbf{1}$, $E(V_1)E(V_2)=E(V_1\cap V_2)$ for any Borel sets V_1,V_2. In addition,

$$E\big((a,b)\big) = E_{b-0} - E_a, \qquad E\big([a,b]\big) = E_b - E_{a-0}$$

and therefore $E\big(\{a\}\big) = E_a - E_{a-0}$.

Definition 1.7.3. *The map* $E:\mathcal{A}_B\to\mathcal{P}(\mathcal{H})$ *defined by* (1.7.1) *is called* the spectral measure *associated with the family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. *This spectral measure is* bounded from below *if there exists* $\lambda_-\in\mathbb{R}$ *such that* $E_\lambda=\mathbf{0}$ *for all* $\lambda<\lambda_-$.

Let us note that for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and any $f\in\mathcal{H}$ one can set

$$F_f(\lambda) := \|E_\lambda f\|^2 = \langle E_\lambda f, f\rangle.$$

Then, one easily checks that the function F_f satisfies the conditions (i)-(iii) of the beginning of Section 1.7.1. Thus, one can associate with each element $f\in\mathcal{H}$ a finite Stieltjes measure m_f on $\mathbb{R}$ which satisfies $m_f(V) = \|E(V)f\|^2 = \langle E(V)f,f\rangle$ for any $V\in\mathcal{A}_B$.

Our next aim is to define integrals of the form

$$\int_a^b \varphi(\lambda)\,E(\mathrm{d}\lambda) \tag{1.7.2}$$

for a continuous function $\varphi : [a,b] \to \mathbb{C}$ and for any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. Such integrals can be defined in the sense of Riemann-Stieltjes by first considering a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$ and a collection $\{y_j\}$ with $y_j \in (x_{j-1}, x_j)$ and by defining the operator

$$\sum_{j=1}^{n} \varphi(y_j) E\big((x_{j-1}, x_j]\big). \tag{1.7.3}$$

It turns out that by considering finer and finer partitions of $[a,b]$, the corresponding expression (1.7.3) strongly converges to an element of $\mathscr{B}(\mathcal{H})$ which is independent of the successive choice of partitions. The resulting operator is denoted by (1.7.2).

Proposition 1.7.4 (Spectral integrals)**.** *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be a spectral family, let $-\infty < a < b < \infty$ and let $\varphi : [a,b] \to \mathbb{C}$ be continuous. Then one has*

(i) $\left\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\right\| = \sup_{\mu\in[a,b]\cap \mathrm{supp}\{E_\lambda\}} |\varphi(\mu)|$,

(ii) $\left(\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\right)^* = \int_a^b \overline{\varphi}(\lambda)\, E(\mathrm{d}\lambda)$,

(iii) For any $f \in \mathcal{H}$, $\left\|\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda) f\right\|^2 = \int_a^b |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda)$,

(iv) If $\psi : [a,b] \to \mathbb{C}$ is continuous, then

$$\int_a^b \varphi(\lambda)\, E(\mathrm{d}\lambda)\ \cdot \int_a^b \psi(\lambda)\, E(\mathrm{d}\lambda) = \int_a^b \varphi(\lambda)\, \psi(\lambda)\, E(\mathrm{d}\lambda).$$

Let us now observe that if the support $\mathrm{supp}\{E_\lambda\}$ is bounded, then one can consider

$$\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda) = s - \lim_{M\to\infty} \int_{-M}^{M} \varphi(\lambda)\, E(\mathrm{d}\lambda). \tag{1.7.4}$$

Similarly, by taking property (iii) of the previous proposition into account, one observes that this limit can also be taken if $\varphi \in L^\infty(\mathbb{R}, \mathbb{C})$. On the other hand, if φ is not bounded on $\mathbb{R}$, the r.h.s. of (1.7.4) is not necessarily well defined. In fact, if φ is not bounded on $\mathbb{R}$ and if $\mathrm{supp}\{E_\lambda\}$ is not bounded either, then the r.h.s. of (1.7.4) is an unbounded operator and can only be defined on a dense domain of $\mathcal{H}$.

Lemma 1.7.5. *Let $\varphi : \mathbb{R} \to \mathbb{C}$ be continuous, and let us set*

$$\mathsf{D}_\varphi := \Big\{ f \in \mathcal{H} \mid \int_{-\infty}^{\infty} |\varphi(\lambda)|^2\, m_f(\mathrm{d}\lambda) < \infty \Big\}.$$

Then the pair $\left(\int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda), \mathsf{D}_\varphi\right)$ defines a densely defined linear operator on $\mathcal{H}$. This operator is self-adjoint if and only if φ is a real function.

A function φ of special interest is the function defined by the identity function id, namely $\mathrm{id}(\lambda) = \lambda$.

Definition 1.7.6. *For any spectral family* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$*, the operator* $\Big(\int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda), \mathsf{D}_{\mathrm{id}}\Big)$ *with*

$$\mathsf{D}_{\mathrm{id}} := \Big\{f \in \mathcal{H} \mid \int_{-\infty}^{\infty} \lambda^2\, m_f(\mathrm{d}\lambda) < \infty\Big\}$$

is called the self-adjoint operator associated with $\{E_\lambda\}$*.*

By this procedure, any spectral family defines a self-adjoint operator on $\mathcal{H}$. *The spectral Theorem* corresponds to the converse statement:

Theorem 1.7.7 (Spectral Theorem)**.** *With any self-adjoint operator* $(A, \mathsf{D}(A))$ *on a Hilbert space* $\mathcal{H}$ *one can associate a unique spectral family* $\{E_\lambda\}$*, called* the spectral family of A*, such that* $\mathsf{D}(A) = \mathsf{D}_{\mathrm{id}}$ *and* $A = \int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda)$*.*

In summary, there is a bijective correspondence between self-adjoint operators and spectral families. This theorem extends the fact that any $n \times n$ hermitian matrix is diagonalizable. The proof of this theorem is not trivial and is rather lengthy. In the sequel, we shall assume it, and state various consequences of this theorem.

Extension 1.7.8. *Study the proof the Spectral Theorem, starting with the version for bounded self-adjoint operators.*

1.7.3 Bounded functional calculus

Let A be a self-adjoint operator in $\mathcal{H}$ and $\{E_\lambda\}$ be the corresponding spectral family.

Definition 1.7.9. *For any bounded and continuous function* $\varphi : \mathbb{R} \to \mathbb{C}$ *one sets* $\varphi(A) \in \mathscr{B}(\mathcal{H})$ *for the operator defined by*

$$\varphi(A) := \int_{-\infty}^{\infty} \varphi(\lambda)\, E(\mathrm{d}\lambda).$$

Exercise 1.7.10. *Show the following equality:* $\mathrm{supp}\{E_\lambda\} = \sigma(A)$*. Note that part of the proof consists in showing that if* $\varphi_z(\lambda) = (\lambda - z)^{-1}$ *for some* $z \in \rho(A)$*, then* $\varphi_z(A) = (A - z)^{-1}$*, where the r.h.s. has been defined in Section 1.6.*

For the next statement, we set $C_b(\mathbb{R})$ for the set of all continuous and bounded complex functions on $\mathbb{R}$.

Proposition 1.7.11. *a) For any* $\varphi \in C_b(\mathbb{R})$ *one has*

(i) $\varphi(A) \in \mathscr{B}(\mathcal{H})$ *and* $\|\varphi(A)\| = \sup_{\lambda\in\sigma(A)} |\varphi(\lambda)|$,

(ii) $\varphi(A)^* = \overline{\varphi}(A)$*, and* $\varphi(A)$ *is self-adjoint if and only if* φ *is real,*

(iii) $\varphi(A)$ *is unitary if and only if* $|\varphi(\lambda)| = 1$.

b) The map $C_b(\mathbb{R}) \ni \varphi \mapsto \varphi(A) \in \mathscr{B}(\mathcal{H})$ *is a* $*$*-homomorphism.*

In the point (iii) above, one can consider the function $\varphi_t \in C_b(\mathbb{R})$ defined by $\varphi_t(\lambda) := e^{-it\lambda}$ for any fixed $t \in \mathbb{R}$. Then, if one sets $U_t := \varphi_t(A)$ one can observe that $U_t U_s = U_{t+s}$ and that the map $\mathbb{R} \ni t \mapsto U_t \in \mathscr{B}(\mathcal{H})$ is strongly continuous. Such a family $\{U_t\}_{t\in\mathbb{R}}$ is called *a strongly continuous unitary group.*

Theorem 1.7.12 (Stone Theorem)**.** *There exists a bijective correspondence between self-adjoint operators on* $\mathcal{H}$ *and strongly continuous unitary groups on* $\mathcal{H}$. *More precisely, if* A *is a self-adjoint operator on* $\mathcal{H}$, *then* $\{e^{-itA}\}_{t\in\mathbb{R}}$ *is a strongly continuous unitary group, while if* $\{U_t\}_{t\in\mathbb{R}}$ *is a strongly continuous unitary group, one sets*

$$\mathsf{D}(A) := \Big\{ f \in \mathcal{H} \mid \exists\, s - \lim_{t\to 0} \frac{1}{t}[U_t - 1]f \Big\}$$

and for $f \in \mathsf{D}(A)$ *one sets* $Af = s - \lim_{t\to 0} \frac{i}{t}[U_t - 1]f$.

Remark 1.7.13. *If the inverse Fourier transform* $\check{\varphi}$ *of* φ *belongs to* $L^1(\mathbb{R})$, *then the following equality holds*

$$\varphi(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \check{\varphi}(t)\, e^{-itA}\, \mathrm{d}t.$$

1.7.4 Spectral parts of a self-adjoint operator

In this section, we consider a fixed self-adjoint operator A (and its associated spectral family $\{E_\lambda\}$), and show that there exists a natural decomposition of the Hilbert space $\mathcal{H}$ with respect to this operator. First of all, recall from Lemma 1.6.6 that the spectrum of any self-adjoint operator is real. In addition, let us recall that for any $\mu \in \mathbb{R}$, one has

$$\mathsf{Ran}\big(E(\{\mu\})\big) = \{f \in \mathcal{H} \mid E(\{\mu\})f = f\}.$$

Then, one observes that the following equivalence holds:

$$f \in \mathsf{Ran}\big(E(\{\mu\})\big) \Longleftrightarrow f \in \mathsf{D}(A) \ \text{ with } Af = \mu f.$$

Indeed, this can be inferred from the equality

$$\|Af - \mu f\|^2 = \int_{-\infty}^{\infty} |\lambda - \mu|^2\, m_f(\mathrm{d}\lambda)$$

which itself can be deduced from the point (iii) of Proposition 1.7.4. Indeed, since the integrand is strictly positive for each $\lambda \neq \mu$, one can have $\|Af - \mu f\| = 0$ if and only if $m_f(V) = 0$ for any Borel set V on $\mathbb{R}$ with $\mu \notin V$. In other words, the measure m_f is supported only on $\{\mu\}$.

Definition 1.7.14. *The set of all* $\mu \in \mathbb{R}$ *such that* $\mathsf{Ran}\,\big(E(\{\mu\})\big) \neq 0$ *is called* the point spectrum of A *or* the set of eigenvalues of A. *One then sets*

$$\mathcal{H}_p(A) := \bigoplus \mathsf{Ran}\,\big(E(\{\mu\})\big)$$

where the sum extends over all eigenvalues of A.

In accordance with what has been presented in Theorem 1.7.1, we define two additional subspaces of $\mathcal{H}$.

Definition 1.7.15.

$$\begin{aligned}\mathcal{H}_{ac}(A) &:= \big\{f \in \mathcal{H} \mid m_f \textit{ is an absolutely continuous measure}\big\}\\ &= \big\{f \in \mathcal{H} \mid \textit{ the function } \lambda \mapsto \|E_\lambda f\|^2 \textit{ is absolutely continuous}\big\},\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{sc}(A) &:= \big\{f \in \mathcal{H} \mid m_f \textit{ is a singular continuous measure}\big\}\\ &= \big\{f \in \mathcal{H} \mid \textit{ the function } \lambda \mapsto \|E_\lambda f\|^2 \textit{ is singular continuous}\big\},\end{aligned}$$

for which the comparison measure is always the Lebesgue measure on $\mathbb{R}$.

Theorem 1.7.16. *Let* A *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$.

a) This Hilbert space can be decomposed as follows

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A),$$

and the restriction of the operator A *to one of these subspaces defines a self-adjoint operator denoted respectively by* A_p, A_{ac} *and* A_{sc}.

b) For any $\varphi \in C_b(\mathbb{R})$, *one has the decomposition*

$$\varphi(A) = \varphi(A_p) \oplus \varphi(A_{ac}) \oplus \varphi(A_{sc}).$$

Moreover, the following equality holds

$$\sigma(A) = \sigma(A_p) \cup \sigma(A_{ac}) \cup \sigma(A_{sc}).$$

Note that one often writes $E_p(A)$, $E_{ac}(A)$ and $E_{sc}(A)$ for the orthogonal projection on $\mathcal{H}_p(A)$, $\mathcal{H}_{ac}(A)$ and $\mathcal{H}_{sc}(A)$, respectively, and with these notations one has $A_p = AE_p(A)$, $A_{ac} = AE_{ac}(A)$ and $A_{sc} = AE_{sc}(A)$. In addition, note that the relation between the set of eigenvalues $\sigma_p(A)$ introduced in Definition 1.6.1 and the set $\sigma(A_p)$ is

$$\sigma(A_p) = \overline{\sigma_p(A)}.$$

Two additional sets are often introduced in relation with the spectrum of A, namely $\sigma_d(A)$ and $\sigma_{ess}(A)$.

Definition 1.7.17. *An eigenvalue* λ *belongs to* the discrete spectrum $\sigma_d(A)$ of A *if and only if* $\mathsf{Ran}\,\big(E(\{\lambda\})\big)$ *is of finite dimension, and* λ *is isolated from the rest of the spectrum of* A*.* The essential spectrum $\sigma_{ess}(A)$ of A *is the complementary set of* $\sigma_d(A)$ *in* $\sigma(A)$*, or more precisely*

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

We end this section with an other characterization of the spectrum of the operator A.

Proposition 1.7.18 (Weyl's criterion)**.** *Let* A *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$*.*

a) A real number λ *belongs to* $\sigma(A)$ *if and only if there exists a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ *such that* $\|f_n\| = 1$ *and* $s - \lim_{n\to\infty}(A-\lambda)f_n = 0$*.*

b) A real number λ *belongs to* $\sigma_{ess}(A)$ *if and only if there exists a sequence* $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ *such that* $\|f_n\| = 1$*,* $w - \lim_{n\to\infty} f_n = 0$ *and* $s - \lim_{n\to\infty}(A-\lambda)f_n = 0$*.*

Chapter 2

C^*-algebras

This chapter is mainly based on the first chapters of the book [Mur90]. Material borrowed from other references will be specified.

2.1 Banach algebras

Definition 2.1.1. A Banach algebra $\mathscr{C}$ *is a complex vector space endowed with an associative multiplication and with a norm* $\|\cdot\|$ *which satisfy for any* $A, B, C \in \mathscr{C}$ *and* $\alpha \in \mathbb{C}$

(i) $(\alpha A)B = \alpha(AB) = A(\alpha B)$,

(ii) $A(B + C) = AB + AC$ *and* $(A + B)C = AC + BC$,

(iii) $\|AB\| \leq \|A\| \, \|B\|$ *(submultiplicativity)*

(iv) $\mathscr{C}$ *is complete with the norm* $\|\cdot\|$.

One says that $\mathscr{C}$ is *abelian* or *commutative* if $AB = BA$ for all $A, B \in \mathscr{C}$. One also says that $\mathscr{C}$ is *unital* if $\mathbf{1} \in \mathscr{C}$, *i.e.* if there exists an element $\mathbf{1} \in \mathscr{C}$ with $\|\mathbf{1}\| = 1$ such that $\mathbf{1}B = B = B\mathbf{1}$ for all $B \in \mathscr{C}$. *A subalgebra* $\mathscr{J}$ *of* $\mathscr{C}$ is a vector subspace which is stable for the multiplication. If $\mathscr{J}$ is norm closed, it is a Banach algebra in itself.

Examples 2.1.2. *(i)* $\mathbb{C}$, $M_n(\mathbb{C})$, $\mathscr{B}(\mathcal{H})$, $\mathscr{K}(\mathcal{H})$ *are Banach algebras, where* $M_n(\mathbb{C})$ *denotes the set of* $n \times n$*-matrices over* $\mathbb{C}$. *All except* $\mathscr{K}(\mathcal{H})$ *are unital, and* $\mathscr{K}(\mathcal{H})$ *is unital if* $\mathcal{H}$ *is finite dimensional.*

(ii) If Ω *is a locally compact topological space,* $C_0(\Omega)$ *and* $C_b(\Omega)$ *are abelian Banach algebras, where* $C_b(\Omega)$ *denotes the set of all bounded and continuous complex functions from* Ω *to* $\mathbb{C}$, *and* $C_0(\Omega)$ *denotes the subset of* $C_b(\Omega)$ *of functions* f *which vanish at infinity,* i.e. *for any* $\varepsilon > 0$ *there exists a compact set* $K \subset \Omega$ *such that* $\sup_{x \in \Omega \setminus K} |f(x)| \leq \varepsilon$. *These algebras are endowed with the* L^∞*-norm, namely* $\|f\| = \sup_{x \in \Omega} |f(x)|$. *Note that* $C_b(\Omega)$ *is unital, while* $C_0(\Omega)$ *is not, except if* Ω *is compact. In this case, one has* $C_0(\Omega) = C(\Omega) = C_b(\Omega)$.

(iii) *If (Ω, μ) is a measure space, then $L^\infty(\Omega)$, the (equivalent classes of) essentially bounded complex functions on Ω is a unital abelian Banach algebra with the essential supremum norm $\|\cdot\|_\infty$.*

(iv) *For any $n \in \mathbb{N}$, the set $BC_u(\mathbb{R}^d)$ of bounded and uniformly continuous complex functions on $\mathbb{R}^d$ is a unital abelian Banach algebra. Recall that $f : \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in \mathbb{R}^d$ with $|x - y| \leq \delta$ one has $|f(x) - f(y)| \leq \varepsilon$. Note that this property can be defined not only on $\mathbb{R}^d$ but on all* uniform spaces.

If S is a subset of a Banach algebra $\mathscr{C}$, the smallest closed subalgebra of $\mathscr{C}$ which contains S is called *the closed algebra generated by S*.

Definition 2.1.3. An ideal *in a Banach algebra $\mathscr{C}$ is a (non-trivial) subalgebra $\mathscr{J}$ of $\mathscr{C}$ such that $AB \in \mathscr{J}$ and $BA \in \mathscr{J}$ whenever $A \in \mathscr{J}$ and $B \in \mathscr{C}$. An ideal $\mathscr{J}$ is* maximal *in $\mathscr{C}$ if $\mathscr{J}$ is proper ($\Leftrightarrow$ not equal to $\mathscr{C}$) and $\mathscr{J}$ is not contained in any other proper ideal of $\mathscr{C}$.*

In the examples presented above, $C_0(\Omega)$ is an ideal of $C_b(\Omega)$, while $\mathscr{K}(\mathcal{H})$ is an ideal of $\mathscr{B}(\mathcal{H})$.

Lemma 2.1.4. *If $\mathscr{C}$ is a Banach algebra and $\mathscr{J}$ is a closed ideal in $\mathscr{C}$, the quotient $\mathscr{C}/\mathscr{J}$ of $\mathscr{C}$ by $\mathscr{J}$, endowed with the multiplication $(A + \mathscr{J})(B + \mathscr{J}) = (AB + \mathscr{J})$ and with the quotient norm $\|A + \mathscr{J}\| := \inf_{B \in \mathscr{J}} \|A + B\|$, is a Banach algebra.*

Proof. The algebraic properties of the quotient are easily verified, and the submultiplicativity is shown below. The completeness of the quotient with respect to the norm is a standard result of normed vector spaces, see for example [Ped89, Prop. 2.1.5].

Let $\varepsilon > 0$ and let $A, B \in \mathscr{C}$. Then

$$\|A + A'\| < \|A + \mathscr{J}\| + \varepsilon \qquad \|B + B'\| < \|B + \mathscr{J}\| + \varepsilon$$

for some $A', B' \in \mathscr{J}$. Hence, by setting $C := A'B + AB' + A'B' \in \mathscr{J}$ one has

$$\|AB + C\| \leq \|A + A'\| \, \|B + B'\| \leq \big(\|A + \mathscr{J}\| + \varepsilon\big)\big(\|B + \mathscr{J}\| + \varepsilon\big).$$

Thus, $\|AB + \mathscr{J}\| \leq \big(\|A + \mathscr{J}\| + \varepsilon\big)\big(\|B + \mathscr{J}\| + \varepsilon\big)$. By letting then $\varepsilon \searrow 0$, we get $\|AB + \mathscr{J}\| \leq \|A + \mathscr{J}\| \, \|B + \mathscr{J}\|$, which corresponds to the submultiplicativity of the quotient norm. □

Definition 2.1.5. A homomorphism *φ between two Banach algebras $\mathscr{C}$ and $\mathscr{Q}$ is a linear map $\varphi : \mathscr{C} \to \mathscr{Q}$ which satisfies $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathscr{C}$. If $\mathscr{C}$ and $\mathscr{Q}$ are unital and if $\varphi(\mathbf{1}) = \mathbf{1}$, one says that φ is* unit preserving *or* a unital homomorphism.

It is easily seen that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a homomorphism, its kernel $\mathsf{Ker}(\varphi)$ is an ideal in $\mathscr{C}$ and its range $\varphi(\mathscr{C})$ is a subalgebra of $\mathscr{Q}$. Alternatively, if $\mathscr{J}$ is an ideal in a Banach algebra $\mathscr{C}$, then *the quotient map* $q : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ is a homomorphism.

Let us now consider an arbitrary unital Banach algebra $\mathscr{C}$, and let $A \in \mathscr{C}$. One says that A is *invertible* if there exists $B \in \mathscr{C}$ such that $AB = \mathbf{1} = BA$. In this case, the element B is denoted by A^{-1} and is called *the inverse of* A. The set of all invertible elements in a unital Banach algebra $\mathscr{C}$ is denoted by $\mathsf{Inv}(\mathscr{C})$.

Exercise 2.1.6. *By using the Neumann series, show that $\mathsf{Inv}(\mathscr{C})$ is an open set in a unital Banach algebra $\mathscr{C}$, and that the map $\mathsf{Inv}(\mathscr{C}) \ni A \mapsto A^{-1} \in \mathscr{C}$ is differentiable.*

On the other hand, let us show that maximal ideals in a unital Banach algebra $\mathscr{C}$ are closed. For this, observe first that for every ideal $\mathscr{J} \neq \mathscr{C}$ we have $\mathscr{J} \cap \mathsf{Inv}(\mathscr{C}) = \emptyset$. Indeed, if one has $A \in \mathscr{J} \cap \mathsf{Inv}(\mathscr{C})$, then for any $B \in \mathscr{C} \setminus \mathscr{J}$ one would have $B = A(A^{-1}B) \in \mathscr{J}$, which is absurd. As a consequence, it follows that $\|\mathbf{1} - A\| \geq 1$ since otherwise A would be invertible with the Neumann series. Consequently, $\mathscr{J}$ can not be dense in $\mathscr{C}$, and thus the closure $\overline{\mathscr{J}}$ of $\mathscr{J}$ is a proper and closed ideal in $\mathscr{C}$. One infers from this that any maximal ideal in $\mathscr{C}$ is closed.

2.2 Spectral theory

The main notions of spectral theory introduced before in the context of $\mathscr{B}(\mathcal{H})$ can be generalized to arbitrary unital Banach algebra.

For any A in a unital Banach algebra $\mathscr{C}$ we define *the spectrum $\sigma_{\mathscr{C}}(A)$ of A with respect to $\mathscr{C}$* by

$$\sigma_{\mathscr{C}}(A) := \big\{ z \in \mathbb{C} \mid (A - z) \notin \mathsf{Inv}(\mathscr{C}) \big\}. \tag{2.2.1}$$

Note that the spectrum $\sigma_{\mathscr{C}}(A)$ of A is never empty, see for example [Mur90, Thm. 1.2.5]. This result is not completely trivial and its proof is based on Liouville's Theorem in complex analysis.

Based on this observation, we state two results which are often quite useful.

Theorem 2.2.1 (Gelfand-Mazur). *If $\mathscr{C}$ is a unital Banach algebra in which every non-zero element is invertible, then $\mathscr{C} = \mathbb{C}\mathbf{1}$.*

Proof. We know from the observation made above that for any $A \in \mathscr{C}$, there exists $z \in \mathbb{C}$ such that $A - z \equiv A - z\mathbf{1} \notin \mathsf{Inv}(\mathscr{C})$. By assumption, it follows that $A = z\mathbf{1}$. □

Lemma 2.2.2. *Let $\mathscr{J}$ be a maximal ideal of a unital abelian Banach algebra $\mathscr{C}$, then $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$.*

Proof. As seen in Lemma 2.1.4, $\mathscr{C}/\mathscr{J}$ is a Banach algebra with unit $\mathbf{1}+\mathscr{J}$; the quotient map $\mathscr{C} \to \mathscr{C}/\mathscr{J}$ is denoted by q. If $\mathscr{I}$ is an ideal in $\mathscr{C}/\mathscr{J}$, then $q^{-1}(\mathscr{I})$ is an ideal of $\mathscr{C}$ containing $\mathscr{J}$, which is therefore either equal to $\mathscr{C}$ or to $\mathscr{J}$, by the maximality of $\mathscr{J}$. Consequently, $\mathscr{I}$ is either equal to $\mathscr{C}/\mathscr{J}$ or to $\mathbf{0}$, and $\mathscr{C}/\mathscr{J}$ has no proper ideal.

Now, if $A \in \mathscr{C}/\mathscr{J}$ and $A \neq \mathbf{0}$, then $A \in \mathsf{Inv}\big(\mathscr{C}/\mathscr{J}\big)$, since otherwise $A(\mathscr{C}/\mathscr{J})$ would be a proper ideal of $\mathscr{C}/\mathscr{J}$. In other words, one has obtained that any non-zero element of $\mathscr{C}/\mathscr{J}$ is invertible, which implies that $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$, by Theorem 2.2.1. □

Lemma 2.2.3. *Let $\mathscr{C}$ be a unital Banach algebra and let $A \in \mathscr{C}$. Then $\sigma_{\mathscr{C}}(A)$ is a closed subset of the disc in the complex plane, centered at 0 and of radius $\|A\|$.*

Proof. If $|z| > \|A\|$, then $\|z^{-1}A\| < 1$, and therefore $(\mathbf{1} - z^{-1}A)$ is invertible (use the Neumann series). Equivalently, this means that $(z - A)$ is invertible, and therefore $z \notin \sigma_{\mathscr{C}}(A)$. Thus, one has obtained that if $z \in \sigma_{\mathscr{C}}(A)$, then $|z| \leq \|A\|$.

Since $\mathsf{Inv}(\mathscr{C})$ is an open set in $\mathscr{C}$, one easily infers that $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ is an open set in $\mathbb{C}$, which means that $\sigma_{\mathscr{C}}(A)$ is a closed set in $\mathbb{C}$. □

Another notion related to the spectrum of A is sometimes convenient. If A belongs to a unital Banach algebra $\mathscr{C}$, its *spectral radius* $r(A)$ is defined by

$$r(A) := \sup_{z \in \sigma_{\mathscr{C}}(A)} |z|.$$

Clearly, it follows from the previous lemma that $r(A) \leq \|A\|$. In addition, the following property holds:

Theorem 2.2.4 (Beurling)**.** *If A is an element of a unital Banach algebra, then*

$$r(A) = \inf_{n \geq 1} \|A^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n}.$$

Proof. See [Mur90, Thm. 1.2.7] or [Ped89, Thm. 4.1.13]. □

For the next statement, recall that if K is a non-empty compact set in $\mathbb{C}$, it complement $\mathbb{C} \setminus K$ admits exactly one unbounded component, and that the bounded components of $\mathbb{C} \setminus K$ are called the *holes* of K.

Proposition 2.2.5. *Let $\mathscr{C}$ be a closed subalgebra of a unital Banach algebra $\mathscr{A}$ which contains the unit of $\mathscr{A}$. Then,*

(i) The set $\mathsf{Inv}(\mathscr{C})$ is a clopen ($\Leftrightarrow$ open and closed) subset of $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$,

(ii) For each $A \in \mathscr{C}$,

$$\sigma_{\mathscr{A}}(A) \subseteq \sigma_{\mathscr{C}}(A) \qquad \text{and} \qquad \partial\sigma_{\mathscr{C}}(A) \subseteq \partial\sigma_{\mathscr{A}}(A),$$

(iii) If $A \in \mathscr{C}$ and $\sigma_{\mathscr{A}}(A)$ has no hole, then $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$.

Proof. Clearly $\mathsf{Inv}(\mathscr{C})$ is an open set in $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$. To see that it is also closed, let (A_n) be a sequence in $\mathsf{Inv}(\mathscr{C})$ converging to a point $A \in \mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$. Then, from the equality $A_n^{-1} - A^{-1} = A_n^{-1}(A - A_n)A^{-1}$, one infers that (A_n^{-1}) converges to A^{-1} in $\mathscr{A}$, so $A^{-1} \in \mathscr{C}$ (by the completeness of $\mathscr{C}$), which implies that $A \in \mathsf{Inv}(\mathscr{C})$. Hence, $\mathsf{Inv}(\mathscr{C})$ is clopen in $\mathscr{C} \cap \mathsf{Inv}(\mathscr{A})$.

If $A \in \mathscr{C}$, the inclusion $\sigma_{\mathscr{A}}(A) \subseteq \sigma_{\mathscr{C}}(A)$ is immediate from the inclusion $\mathsf{Inv}(\mathscr{C}) \subseteq \mathsf{Inv}(\mathscr{A})$.

If $z \in \partial\sigma_{\mathscr{C}}(A)$, then there is a sequence (z_n) in $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ converging to z. Hence, $(A - z_n) \in \mathsf{Inv}(\mathscr{C})$, and $(A - z) \notin \mathsf{Inv}(\mathscr{C})$, so $(A - z) \notin \mathsf{Inv}(\mathscr{A})$, by the point (i). Also, $A - z_n \in \mathsf{Inv}(\mathscr{A})$, so $z_n \in \mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$. Therefore, $z \in \partial\sigma_{\mathscr{A}}(A)$. This proves the point (ii).

If $A \in \mathscr{C}$ and $\sigma_{\mathscr{A}}(A)$ has no hole, then $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$ is connected. Since $\mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$ is a clopen subset of $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A)$ by the points (i) and (ii), it follows that $\mathbb{C} \setminus \sigma_{\mathscr{A}}(A) = \mathbb{C} \setminus \sigma_{\mathscr{C}}(A)$, and therefore $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$. □

Let us end this section with a construction which can be used if a Banach algebra $\mathscr{C}$ has no unit. Consider the set $\tilde{\mathscr{C}} := \mathscr{C} \oplus \mathbb{C}$ with the multiplication

$$(A, z)(B, y) = (AB + zB + yA, zy).$$

This algebra contains a unit $\mathbf{1} = (\mathbf{0}, 1)$ and is call *a unitization of* $\mathscr{C}$. Clearly, the map $\mathscr{C} \ni A \mapsto (A, 0) \in \tilde{\mathscr{C}}$ is an injective homomorphism, which can be used to identify $\mathscr{C}$ with an ideal of $\tilde{\mathscr{C}}$. It is quite common to write simply $A + z$ for the element (A, z) of $\tilde{\mathscr{C}}$. Endowed with the norm $\|A + z\| := \|A\| + |z|$, $\tilde{\mathscr{C}}$ is a unital Banach algebra, which is abelian if $\mathscr{C}$ is abelian.

If $\mathscr{C}$ is a non-unital Banach algebra and $A \in \mathscr{C}$, one sets $\sigma_{\mathscr{C}}(A) := \sigma_{\tilde{\mathscr{C}}}(A)$.

2.3 The Gelfand representation

In this section, we concentrate on abelian Banach algebras and state a fundamental result for these algebras. First of all, let us observe that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a unital homomorphism between the unital Banach algebras $\mathscr{C}$ and $\mathscr{Q}$, then $\varphi\big(\mathsf{Inv}(\mathscr{C})\big) \subset \mathsf{Inv}(\mathscr{Q})$, and therefore $\sigma_{\mathscr{Q}}(\varphi(A)) \subset \sigma_{\mathscr{C}}(A)$ whenever $A \in \mathscr{C}$.

Definition 2.3.1. A character τ *on an abelian algebra* $\mathscr{C}$ *is a non-zero homomorphism from* $\mathscr{C}$ *to* $\mathbb{C}$. *The set of all characters of* $\mathscr{C}$ *is denoted by* $\Omega(\mathscr{C})$.

Let us immediately observe that if $\tau \in \Omega(\mathscr{C})$ for a unital abelian Banach algebra $\mathscr{C}$, then $\|\tau\| = 1$. Indeed, if $A \in \mathscr{C}$, one has $\tau(A) \subset \sigma_{\mathscr{C}}(A)$, and therefore $|\tau(A)| \leq \|A\|$. Hence $\|\tau\| \leq 1$, but $\tau(\mathbf{1}) = 1$ since $\tau(\mathbf{1}) = \tau(\mathbf{1})^2$ and $\tau(\mathbf{1}) \neq 0$.

For the next statement, we introduce the notation $M(\mathscr{C})$ for the set of maximal ideals of a Banach algebra $\mathscr{C}$.

Proposition 2.3.2. *Let* $\mathscr{C}$ *be a unital abelian Banach algebra. There is a bijection* $\tau \leftrightarrow \mathsf{Ker}(\tau)$ *between the set* $\Omega(\mathscr{C})$ *of characters of* $\mathscr{C}$ *and the set* $M(\mathscr{C})$. *Additionally, for each* $A \in \mathscr{C}$ *one has*

$$\sigma_{\mathscr{C}}(A) = \big\{\tau(A) \mid \tau \in \Omega(\mathscr{C})\big\}.$$

Proof. Let us first take $\mathscr{J} \in M(\mathscr{C})$ and consider the quotient Banach algebra $\mathscr{C}/\mathscr{J}$. By Lemma 2.2.2, it follows that $\mathscr{C}/\mathscr{J} = \mathbb{C}\mathbf{1}$, and therefore the quotient map $\tau : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ belongs $\Omega(\mathscr{C})$. Conversely, if $\tau \in \Omega(\mathscr{C})$, then $\mathsf{Ker}(\tau)$ is an ideal in $\mathscr{C}$. In addition, one has $\mathscr{C} = \mathsf{Ker}(\tau) + \mathbb{C}\mathbf{1}$, since $\big(A - \tau(A)\mathbf{1}\big) \in \mathsf{Ker}(\tau)$. Consequently, $\mathsf{Ker}(\tau)$ is of co-dimension 1, and therefore is maximal.

Now, we show that any $A \in \mathscr{C} \setminus \mathsf{Inv}(\mathscr{C})$ is contained in a maximal ideal. Indeed, one easily observes that $A \in \mathscr{C}A$, with $\mathscr{C}A$ an ideal of $\mathscr{C}$ which does not contain $\mathbf{1}$. Then, the set of ideals that contains A but not $\mathbf{1}$ is inductively ordered by induction (because a union of an increasing family of ideals is an ideal), and a maximal element of this ordering is a maximal ideal. From Zorn's Lemma, it follows that A is contained in a maximal ideal.

Finally, if $A \in \mathscr{C}$ and $z \in \sigma_{\mathscr{C}}(A)$, then $(A - z) \notin \mathsf{Inv}(\mathscr{C})$. Therefore, there exists a character $\tau \in \Omega(\mathscr{C})$ such that $(A-z) \equiv (A - z\mathbf{1})$ belongs to the corresponding maximal ideal $\mathsf{Ker}(\tau)$. Accordingly, $\tau(A - z\mathbf{1}) = 0 \iff \tau(A) = z$. Conversely, if $\tau(A) = z$ for some $\tau \in \Omega(\mathscr{C})$, then $z \in \sigma_{\mathbb{C}}\big(\tau(A)\big) \subset \sigma_{\mathscr{C}}(A)$, by the observation made at the beginning of the section. □

Remark 2.3.3. *In the previous statement, if $\mathscr{C}$ is not unital one has for any $A \in \mathscr{C}$*

$$\sigma_{\mathscr{C}}(A) = \big\{\tau(A) \mid \tau \in \Omega(\mathscr{C})\big\} \cup \{0\}. \tag{2.3.1}$$

Indeed, if $\tau_\infty : \tilde{\mathscr{C}} \to \mathbb{C}$ denotes the character defined by $\tau_\infty(A,z) = z$, then one has $\Omega(\tilde{\mathscr{C}}) = \big\{\tilde{\tau} \mid \tau \in \Omega(\mathscr{C})\big\} \cup \{\tau_\infty\}$ with $\tilde{\tau}(A,z) = \tau(A) + z$, and

$$\sigma_{\mathscr{C}}(A) = \sigma_{\tilde{\mathscr{C}}}(A) = \{\tau(A,0) \mid \tau \in \Omega(\tilde{\mathscr{C}})\} = \{\tau(A) \mid \tau \in \Omega(\mathscr{C})\} \cup \{0\}. \tag{2.3.2}$$

Since for any abelian Banach algebra $\mathscr{C}$, any $A \in \mathscr{C}$ and any $\tau \in \Omega(\mathscr{C})$ one has $|\tau(A)| \leq \|A\|$, it follows that $\Omega(\mathscr{C})$ is contained in the closed unit ball of the dual space $\mathscr{C}^*$. Thus, we can endow $\Omega(\mathscr{C})$ with the relative weak* topology and call the topological space $\Omega(\mathscr{C})$ the *character space*, or *spectrum* of $\mathscr{C}$.

Proposition 2.3.4. *If $\mathscr{C}$ is an abelian Banach algebra, then $\Omega(\mathscr{C})$ is a locally compact Hausdorff*[1] *space. If $\mathscr{C}$ is unital, then $\Omega(\mathscr{C})$ is compact.*

Proof. If $\mathscr{C}$ is unital, then it can be checked that $\Omega(\mathscr{C})$ is weak* closed in the closed unital ball $\mathscr{B}$ of $\mathscr{C}^*$. Since $\mathscr{B}$ is weak* compact (Banach-Alaoglu Theorem), it follows that $\Omega(\mathscr{C})$ is weak* compact.

If $\mathscr{C}$ is not unital, then $\Omega(\mathscr{C}) \cong \Omega(\tilde{\mathscr{C}}) \setminus \{\tau_\infty\}$, and therefore one obtains that $\Omega(\tilde{\mathscr{C}})$ is only locally compact. □

For any A in an abelian algebra $\mathscr{C}$ one defines the function $\hat{A}$ by

$$\hat{A} : \Omega(\mathscr{C}) \ni \tau \mapsto \hat{A}(\tau) \in \mathbb{C}$$

with $\hat{A}(\tau) := \tau(A)$. The topology of $\Omega(\mathscr{C})$ makes this function continuous. In addition, since for any $\varepsilon > 0$ the set $\{\tau \in \Omega(\mathscr{C}) \mid |\tau(A)| \geq \varepsilon\}$ is weak* closed in the closed unit ball of $\mathscr{C}^*$, and weak* compact by the Banach-Alaoglu Theorem, it follows that $\hat{A} \in C_0\big(\Omega(\mathscr{C})\big)$. Note that the map $A \mapsto \hat{A}$ is called *the Gelfand transform.*

[1] A Hausdorff space is a topological space in which distinct points have disjoint neighbourhoods. The weak* topology is Hausdorff.

Theorem 2.3.5. *Let $\mathscr{C}$ be an abelian Banach algebra. Then the map*

$$\mathscr{C} \ni A \mapsto \hat{A} \in C_0\big(\Omega(\mathscr{C})\big)$$

is a norm decreasing homomorphism, and $\|\hat{A}\|_\infty = r(A)$. If $\mathscr{C}$ is unital, then $\sigma_{\mathscr{C}}(A) = \hat{A}\big(\Omega(\mathscr{C})\big)$, while if $\mathscr{C}$ is not unital, $\sigma_{\mathscr{C}}(A) = \hat{A}\big(\Omega(\mathscr{C})\big) \cup \{0\}$, for any $A \in \mathscr{C}$.

Proof. It is easily checked that the mentioned map is a homomorphism. The spectral properties are direct consequences of (2.3.1) and (2.3.2), while the property on the norm follows from the observation that $\|\hat{A}\|_\infty = r(A) \leq \|A\|$. □

Note that the interpretation of the character space as a sort of generalized spectrum is motivated by the following result.

Lemma 2.3.6. *Let $\mathscr{C}$ be a unital Banach algebra, and let $\mathscr{A}$ be the unital subalgebra generated by $\mathbf{1}$ and an element $A \in \mathscr{C}$. Then $\mathscr{A}$ is abelian and the map*

$$\phi_A : \Omega(\mathscr{A}) \to \sigma_{\mathscr{A}}(A), \quad \phi_A(\tau) := \tau(A) \tag{2.3.3}$$

is a homeomorphism.

Proof. It is clear that the algebra $\mathscr{A}$ is abelian, and that ϕ_A is a continuous bijection. Since $\Omega(\mathscr{A})$ and $\sigma_{\mathscr{A}}(A)$ are compact Hausdorff spaces, the map ϕ_A is a homeomorphism (open mapping theorem). □

2.4 Basics on C^*-algebras

Definition 2.4.1. *A* Banach $*$-algebra *or* B^*-algebra *is a Banach algebra $\mathscr{C}$ together with an involution * satisfying for any $A, B \in \mathscr{C}$ and $\alpha \in \mathbb{C}$*

(i) $(A^*)^* = A$,

(ii) $(A + B)^* = A^* + B^*$,

(iii) $(\alpha A)^* = \overline{\alpha} A^*$,

(iv) $(AB)^* = B^* A^*$.

Clearly, if $\mathscr{C}$ is a unital B^*-algebra, then $\mathbf{1}^* = \mathbf{1}$.

Exercise 2.4.2. *Show that $\|A^*\| = \|A\|$ whenever A belongs to a B^*-algebra.*

Definition 2.4.3. A C^*-algebra *is a B^*-algebra $\mathscr{C}$ for which the following additional property is satisfied:*

$$\|A^* A\| = \|A\|^2 \qquad \forall\, A \in \mathscr{C}. \tag{2.4.1}$$

Examples 2.4.4. *All examples mentioned in Examples 2.1.2 are in fact C^*-algebras, once complex conjugation is considered as the involution for complex functions. In addition, let us observe that for a family $\{\mathscr{C}_i\}_{i\in I}$ of C^*-algebras, the direct sum $\oplus_{i\in I}\mathscr{C}_i$, with the pointwise involution and the supremum norm, is also a C^*-algebra.*

Note that a C^*-*subalgebra* of a C^*-algebra $\mathscr{C}$ is a norm closed subalgebra of $\mathscr{C}$ which is stable for the involution. It is clearly a C^*-algebra in itself. Note also that if $\mathscr{C}$ and $\mathscr{Q}$ are C^*-algebras, then $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism if φ is a homomorphism and if in addition $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathscr{C}$. An ideal $\mathscr{J}$ in a C^*-algebra is *self-adjoint* if it is stable for the involution.

Definition 2.4.5. *Let $\mathscr{C}$ be a C^*-algebra. An element $A \in \mathscr{C}$ satisfying $A = A^*$ is called* self-adjoint *or* hermitian*, an element $P \in \mathscr{C}$ satisfying $P = P^2 = P^*$ is called* an orthogonal projection*, and an element $A \in \mathscr{C}$ satisfying $AA^* = A^*A$ is called* a normal element of $\mathscr{C}$*. In addition, if $\mathscr{C}$ is unital, an element $U \in \mathscr{C}$ satisfying $UU^* = \mathbf{1} = U^*U$ is called* a unitary*,*

Note that it then follows from relation (2.4.1) that $\|U\| = 1$ for any unitary in $\mathscr{C}$, and that $\|P\| = 1$ for any (non-trivial) orthogonal projection in $\mathscr{C}$.

For the next statement, let us set

$$\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

Lemma 2.4.6. *Any self-adjoint element A in a unital C^*-algebra $\mathscr{C}$ satisfies $\sigma_{\mathscr{C}}(A) \subset \mathbb{R}$. If U is a unitary element of $\mathscr{C}$, then $\sigma_{\mathscr{C}}(U) \subset \mathbb{T}$.*

Proof. First of all, from the equality $\big((C - z)^{-1}\big)^* = (C^* - \overline{z})^{-1}$, one infers that if $z \in \sigma_{\mathscr{C}}(C)$, then $\overline{z} \in \sigma_{\mathscr{C}}(C^*)$, for any $C \in \mathscr{C}$. Furthermore, from the equality

$$z^{-1}(z - C)C^{-1} = -(z^{-1} - C^{-1}),$$

one also deduces that if $z \in \sigma_{\mathscr{C}}(C)$ for some $C \in \mathsf{Inv}(\mathscr{C})$, then $z^{-1} \in \sigma_{\mathscr{C}}(C^{-1})$.

Now, for a unitary $U \in \mathscr{C}$, one deduces from the above computations that if $z \in \sigma_{\mathscr{C}}(U)$, then $\overline{z}^{-1} \in \sigma_{\mathscr{C}}\big((U^*)^{-1}\big) = \sigma_{\mathscr{C}}(U)$. Since $\|U\| = 1$ one then infers from Lemma 2.2.3 that $|z| \leq 1$ and $|z^{-1}| \leq 1$, which means $z \in \mathbb{T}$.

If $A = A^* \in \mathscr{C}$, one sets $e^{iA} := \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}$ and observes that

$$(e^{iA})^* = e^{-iA} = (e^{iA})^{-1}.$$

Therefore, e^{iA} is a unitary element of $\mathscr{C}$ and it follows that $\sigma_{\mathscr{C}}\big(e^{iA}\big) \subset \mathbb{T}$. Now, let us assume that $z \in \sigma_{\mathscr{C}}(A)$, set $B := \sum_{n=1}^{\infty} \frac{i^n(A-z)^{n-1}}{n!}$, and observe that B commutes with A. Then one has

$$e^{iA} - e^{iz} = (e^{i(A-z)} - 1)e^{iz} = (A - z)Be^{iz}.$$

It follows from this equality that $e^{iz} \in \sigma_{\mathscr{C}}(e^{iA})$. Indeed, if $\big(e^{iA} - e^{iz}\big) \in \mathsf{Inv}(\mathscr{C})$, then $Be^{iz}\big(e^{iA} - e^{iz}\big)^{-1}$ would be an inverse for $(A - z)$, which can not be since $z \in \sigma_{\mathscr{C}}(A)$. From the preliminary computation, one deduces that $|e^{iz}| = 1$, which holds if and only if $z \in \mathbb{R}$. One has thus obtains that $\sigma_{\mathscr{C}}(A) \subset \mathbb{R}$. $\square$

The following statement is an important result for the spectral theory in the framework of C^*-algebras. It shows that the computation of the spectrum does not depend on the surrounding algebra.

Theorem 2.4.7. *Let $\mathscr{C}$ be a C^*-subalgebra of a unital C^*-algebra $\mathscr{A}$ which contains the unit of $\mathscr{A}$. Then for any $A \in \mathscr{C}$,*

$$\sigma_{\mathscr{C}}(A) = \sigma_{\mathscr{A}}(A).$$

Proof. First of all, suppose that A is a self-adjoint element of $\mathscr{C}$. Then, since $\sigma_{\mathscr{A}}(A) \subset \mathbb{R}$, it follows from Proposition 2.2.5.(iii) that $\sigma_{\mathscr{A}}(A) = \sigma_{\mathscr{C}}(A)$. Alternatively, this means that A is invertible in $\mathscr{C}$ if and only if A is invertible in $\mathscr{A}$.

Now suppose that A is an arbitrary element of $\mathscr{C}$ which is invertible in $\mathscr{A}$, *i.e.* there exists $B \in \mathscr{A}$ such that $AB = BA = \mathbf{1}$. Then $A^*B^* = B^*A^* = \mathbf{1}$, so that $AA^*B^*B = \mathbf{1} = B^*BAA^*$, and this means that AA^* is invertible in $\mathscr{A}$, and therefore also in $\mathscr{C}$. Hence, there exists $C \in \mathscr{C}$ such that $AA^*C = \mathbf{1} = CAA^*$. One infers then that $A^*C = B$, which implies that $B \in \mathscr{C}$ and thus that A is invertible in $\mathscr{C}$. As a consequence, for any $A \in \mathscr{C}$ its invertibility in $\mathscr{A}$ is equivalent to its invertibility in $\mathscr{C}$, which directly implies the statement of the theorem. □

Because of the previous result, it is common to denote by $\sigma(A)$ the spectrum of an element A of a C^*-algebra, without specifying in which algebra the spectrum is computed. Let us also mention an additional result concerning the spectral radius:

Exercise 2.4.8. *If A is a self-adjoint element of a C^*-algebra $\mathscr{C}$, show that $r(A) = \|A\|$.*

Let us observe that this simple result has an important corollary:

Corollary 2.4.9. *There is at most one norm on a $*$-algebra making it a C^*-algebra.*

Proof. If $\|\cdot\|_1$, $\|\cdot\|_2$ are norms on a $*$-algebra $\mathscr{C}$ making it a C^*-algebra, then for any $A \in \mathscr{C}$ one has $\|A\|_j^2 = \|A^*A\|_j = r(A^*A)$, and therefore $\|A\|_1 = \|A\|_2$. □

We have already seen at the end of Section 2.2 how we can construct a unital Banach algebra $\tilde{\mathscr{C}}$ from a non-unital Banach algebra $\mathscr{C}$. However, if $\mathscr{C}$ is a C^*-algebra, the resulting algebra $\tilde{\mathscr{C}}$ is not a C^*-algebra in general. We shall now see how the construction can be adapted.

A double centralizer for a C^*-algebra $\mathscr{C}$ is a pair (L, R) of bounded linear maps on $\mathscr{C}$ such that for all $A, B \in \mathscr{C}$ one has

$$L(AB) = L(A)\,B, \quad R(AB) = A\,R(B), \quad \text{and} \quad R(A)\,B = A\,L(B).$$

For example, if $C \in \mathscr{C}$, then one can define a double centralizer (L_C, R_C) by $L_C(A) := CA$ and $R_C(A) := AC$. One then easily checks that

$$\|C\| = \sup_{\|A\| \leq 1} \|CA\| = \sup_{\|A\| \leq 1} \|AC\|,$$

and therefore $\|L_C\| = \|R_C\| = \|C\|$.

More generally one has:

Exercise 2.4.10. *If (L, R) is a double centralizer for a C^*-algebra, show that $\|L\| = \|R\|$.*

Thus, for any C^*-algebra $\mathscr{C}$, one denotes by $\mathscr{M}(\mathscr{C})$ the set of double centralizers of $\mathscr{C}$ and endows it with the norm $\|(L,R)\| := \|R\| = \|L\|$. $\mathscr{M}(\mathscr{C})$ becomes then a closed vector subspace of $\mathscr{B}(\mathscr{C}) \oplus \mathscr{B}(\mathscr{C})$. If in addition, one endows this set with the multiplication

$$(L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1)$$

and with the involution $(L,R)^* = (R^*, L^*)$ with $L^*(A) = \big(L(A^*)\big)^*$ and $R^*(A) = \big(R(A^*)\big)^*$, then one ends up with:

Proposition 2.4.11. *If $\mathscr{C}$ is a C^*-algebra, then $\mathscr{M}(\mathscr{C})$ is also a C^*-algebra.*

Proof. We only prove the property that $\|(L,R)^*(L,R)\| = \|(L,R)\|^2$, the other conditions being quite straightforward. For that purpose, let $A \in \mathscr{C}$ with $\|A\| \leq 1$. Then one has

$$\begin{aligned} \|L(A)\|^2 = \big\|\big(L(A)\big)^* L(A)\big\| = \|L^*(A^*)L(A)\| &= \big\|A\,R^*\big(L(A)\big)\big\| \\ &\leq \|R^* L\| = \big\|(L,R)^*(L,R)\big\|, \end{aligned}$$

which implies that

$$\|(L,R)\|^2 = \sup_{\|A\|\leq 1} \|L(A)\|^2 \leq \|(L,R)^*(L,R)\| \leq \|(L,R)\|^2.$$

One thus infers that $\|(L,R)^*(L,R)\| = \|(L,R)\|^2$. □

The C^*-algebra $\mathscr{M}(\mathscr{C})$ is called *the multiplier algebra* of $\mathscr{C}$, and the map $\mathscr{C} \ni A \mapsto (L_A, R_A) \in \mathscr{M}(\mathscr{C})$ is an isometric $*$-homomorphism of $\mathscr{C}$ into $\mathscr{M}(\mathscr{C})$. We can therefore identify $\mathscr{C}$ with a C^*-subalgebra of $\mathscr{M}(\mathscr{C})$. In fact, $\mathscr{C}$ is an ideal in $\mathscr{M}(\mathscr{C})$, and since $\mathbf{1} \in \mathscr{B}(\mathscr{C})$ the algebra $\mathscr{M}(\mathscr{C})$ is a unital C^*-algebra with unit $(\mathbf{1}, \mathbf{1})$. Note that $\mathscr{C} = \mathscr{M}(\mathscr{C})$ if and only if $\mathscr{C}$ is unital, and that $\mathscr{M}(\mathscr{C})$ is in fact the largest *unitization* of $\mathscr{C}$ in the following sense:

Theorem 2.4.12. *If $\mathscr{J}$ be a closed self-adjoint ideal in a C^*-algebra $\mathscr{C}$, then there exists a unique $*$-homomorphism $\varphi : \mathscr{C} \to \mathscr{M}(\mathscr{J})$ such that φ is the identity map on $\mathscr{J}$. Moreover, φ is injective if and only $\mathscr{J}$ is essential[2] in $\mathscr{C}$.*

Proof. See Proposition 2.2.14 of [W-O93] or Theorem 3.1.8 of [Mur90]. □

Let us recall that a $*$-isomorphism is a bijective $*$-homomorphism. In the next lemma, we deduce a consequence of the previous theorem.

Lemma 2.4.13. *If $\mathscr{C}$ is a C^*-algebra, then there exists a unique norm on its unitization $\tilde{\mathscr{C}}$ making it a C^*-algebra.*

[2] One says that a closed ideal $\mathscr{J}$ in a C^*-algebra $\mathscr{C}$ is *essential* if $AB = \mathbf{0}$ for all $B \in \mathscr{J}$ implies $A = \mathbf{0}$.

Proof. Uniqueness of the norm is given by Corollary 2.4.9. The proof of the existence falls into two cases, depending on whether $\mathscr{C}$ is unital or not.

Let us consider first the case of a unital C^*-algebra $\mathscr{C}$. Then, the map $\varphi : \tilde{\mathscr{C}} \to \mathscr{C} \oplus \mathbb{C}$ defined by $\varphi(A, z) = (A + z\mathbf{1}, z)$ is a $*$-isomorphism. Hence, one gets a C^*-norm on $\tilde{\mathscr{C}}$ by setting $\|(A, z)\| := \|\varphi(A, z)\|$.

Suppose now that $\mathscr{C}$ has no unit. If $\mathbf{1}$ denotes the unit of $\mathscr{M}(\mathscr{C})$, then $\mathscr{C} \cap \mathbb{C}\mathbf{1} = 0$. The map φ from $\tilde{\mathscr{C}}$ to the subalgebra $\mathscr{C} \oplus \mathbb{C}\mathbf{1}$ of $\mathscr{M}(\mathscr{C})$ defined by $\varphi(A, z) = A + z\mathbf{1}$ is a $*$-isomorphism, so we get a C^*-norm on $\tilde{\mathscr{C}}$ by setting $\|(A, z)\| := \|\varphi(A, z)\|$. □

From now on, we shall always consider the unitization $\tilde{\mathscr{C}}$ of a C^*-algebra endowed with its C^*-norm. Note in addition, that $\mathscr{M}(\mathscr{C})$ is usually much bigger than $\tilde{\mathscr{C}}$. For example, if $\mathscr{C} = C_0(\Omega)$ for a locally compact space Ω, then $\mathscr{M}(\mathscr{C}) = C_b(\Omega)$.

It is easily observed that if $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism between $*$-algebras, then φ extends uniquely to a unital $*$-homomorphism $\tilde{\varphi} : \tilde{\mathscr{C}} \to \tilde{\mathscr{Q}}$.

Lemma 2.4.14. *A $*$-homomorphism $\varphi : \mathscr{C} \to \mathscr{Q}$ from a B^*-algebra $\mathscr{C}$ to a C^*-algebra $\mathscr{Q}$ is necessarily norm decreasing.*

Proof. Without lost of generality, one can consider $\mathscr{C}$ and $\mathscr{Q}$ unital (by going to $\tilde{\mathscr{C}}$ and $\tilde{\mathscr{Q}}$ if necessary). For $A \in \mathscr{C}$ one has $\sigma_{\mathscr{Q}}\big(\varphi(A)\big) \subset \sigma_{\mathscr{C}}(A)$, and therefore

$$\|\varphi(A)\|^2 = \big\|\varphi(A)^*\varphi(A)\big\| = \|\varphi(A^*A)\| = r\big(\varphi(A^*A)\big) \leq r(A^*A) \leq \|A^*A\| \leq \|A\|^2.$$

It thus follows that $\|\varphi(A)\| \leq \|A\|$. □

Let us observe that an important corollary can be deduced from the previous lemma, namely any $*$-isomorphism between C^*-algebras is necessarily isometric.

Our next aim is to show that the Gelfand representation contained in Theorem 2.3.5 can be improved in the context of abelian C^*-algebras. For that purpose, observe first that any character on a C^*-algebra preserves adjoints. Indeed, let $\mathscr{C}$ be a C^*-algebra and let τ be a character on $\mathscr{C}$. Then, for any $A \in \mathscr{C}$, let us set $A = \Re(A) + i\Im(A)$ (with $\Re(A) := \frac{A+A^*}{2}$ and $\Im(A) := \frac{A-A^*}{2i}$ self-adjoint) and observe that

$$\tau(A^*) = \tau\big(\Re(A) - i\Im(A)\big) = \tau\big(\Re(A)\big) - i\tau\big(\Im(A)\big) = \overline{\tau\big(\Re(A) + i\Im(A)\big)} = \overline{\tau(A)}.$$

Theorem 2.4.15 (Gelfand representation)**.** *For any non-zero abelian C^*-algebra $\mathscr{C}$, the Gelfand representation*

$$\mathscr{C} \ni A \mapsto \hat{A} \in C_0\big(\Omega(\mathscr{C})\big) \tag{2.4.2}$$

is an isometric $*$-isomorphism.

Proof. Let us denote by φ the homomorphism defined in (2.4.2). It follows from Theorem 2.3.5 that φ is a norm decreasing homomorphism, with $\|\hat{A}\| = r(A)$, for any $A \in \mathscr{C}$. Now, if $\tau \in \Omega(\mathscr{C})$ one has $[\varphi(A^*)](\tau) = \tau(A^*) = \overline{\tau(A)} = \overline{[\varphi(A)](\tau)} = [\varphi(A)^*](\tau)$, which means that φ is a $*$-homomorphism. Moreover, φ is an isometry since

$$\|\varphi(A)\|^2 = \|\varphi(A)^*\varphi(A)\| = \|\varphi(A^*A)\| = r(A^*A) = \|A^*A\| = \|A\|^2.$$

Then, $\varphi(\mathscr{C})$ is a closed $*$-subalgebra of $C_0\big(\Omega(\mathscr{C})\big)$ separating the points of $\Omega(\mathscr{C})$, and having the property that for any $\tau \in \Omega(\mathscr{C})$ there is an element $A \in \mathscr{C}$ such that $[\varphi(A)](\tau) = \tau(A) \neq 0$. The Stone-Weierstrass Theorem implies therefore that $\varphi(\mathscr{C}) = C_0\big(\Omega(\mathscr{C})\big)$. □

The following exercise shows the coherence of the theory:

Exercise 2.4.16. *Let $\boldsymbol{\Omega}$ be a compact Hausdorff space, and for each $x \in \boldsymbol{\Omega}$ let τ_x be the character on $C(\boldsymbol{\Omega})$ defined by $\tau_x(f) = f(x)$ for any $f \in C(\boldsymbol{\Omega})$. Show that the map*

$$\boldsymbol{\Omega} \ni x \mapsto \tau_x \in \Omega\big(C(\boldsymbol{\Omega})\big)$$

is a homeomorphism.

The Gelfand representation has various useful applications. One is contained in the proof of the following statement. For this proof, we also need the following observation: If $\phi : \Omega \to \Omega'$ is a continuous map between compact Hausdorff spaces Ω and Ω', then the transpose map:

$$\phi^t : C(\Omega') \to C(\Omega), \qquad \phi^t(f) := f \circ \phi$$

is a unital $*$-homomorphism. Moreover, if ϕ is a homeomorphism, then ϕ^t is a $*$-isomorphism.

Proposition 2.4.17. *Let A be a normal element of a unital C^*-algebra $\mathscr{C}$, and let z be the inclusion map of $\sigma(A)$ in $\mathbb{C}$. Then there exists a unique unital $*$-homomorphism $\varphi : C\big(\sigma(A)\big) \to \mathscr{C}$ such that $\varphi(z) = A$. Moreover, φ is isometric and the image of φ is the C^*-subalgebra of $\mathscr{C}$ generated by A and $\mathbf{1}$.*

Proof. Let $\mathscr{A}$ be the unital C^*-subalgebra of $\mathscr{C}$ generated by A and $\mathbf{1}$, and let $\psi : \mathscr{A} \to C\big(\Omega(\mathscr{A})\big)$ be the Gelfand representation. By Theorem 2.4.15 ψ is a $*$-isomorphism. In addition, we know from Lemma 2.3.6 that the map ϕ_A defined in (2.3.3) is a homeomorphism, and therefore the map $\phi_A^t : C\big(\sigma(A)\big) \to C\big(\Omega(\mathscr{A})\big)$ is also a $*$-isomorphism. It then follows that the composed map $\varphi := \psi^{-1} \circ \phi_A^t : C\big(\sigma(A)\big) \to \mathscr{A}$ is a unital $*$-homomorphism, with $\varphi(z) = A$ since $\varphi(z) = \psi^{-1}\big(\phi_A^t(z)\big) = \psi^{-1}(\hat{A}) = A$. From the Stone-Weierstrass Theorem, we know that $C\big(\sigma(A)\big)$ is generated by 1 and z; φ is therefore the unique unital $*$-homomorphism from $C\big(\sigma(A)\big)$ to $\mathscr{C}$ such that $\varphi(z) = A$.

The remaining part of the proof is rather clear. □

Based on the idea developed in the previous proof, it is natural to set the following definitions: If S is any subset of a C^*-algebra, we denote by $C^*(S)$ the smallest C^*-algebra generated by S. Clearly, $C^*(S) \subset \mathscr{C}$, and $C^*(A) := C^*(\{A\})$ is an abelian algebra if A is normal. If A is self-adjoint, $C^*(A)$ is the closure of the set of polynomials in A with zero constant term. On the other hand, $C^*(\{A, \mathbf{1}\})$ is the closure of the set of polynomials in A with constant terms.

Let us finally mention that a bounded functional calculus similar to the one developed in Section 1.7.3 can also be defined in the C^*-algebraic framework. We mention below a useful result, but refer to [Mur90, Thm. 2.1.14] for its proof.

Theorem 2.4.18 (Spectral mapping)**.** *Let A be a normal element in a unital C^*-algebra $\mathscr{C}$, and let $\varphi \in C\big(\sigma(A)\big)$. Then the following equality holds:*

$$\sigma\big(\varphi(A)\big) = \varphi\big(\sigma(A)\big).$$

Moreover, if $\psi \in C\big(\sigma(\varphi(A))\big)$, then $[\psi \circ \varphi](A) = \psi\big(\varphi(A)\big)$.

2.5 Additional material on C^*-algebras

In this section we add some standard material on C^*-algebras. More information can be found in Chapters 2 and 3 of [Mur90].

Let us first observe that if $\mathscr{C} = C_0(\Omega)$ for a locally compact space Ω, then a natural notion of positivity on $\mathscr{C}$ exists. Indeed, if $\mathscr{C}_{sa}$ denote the subset of $\mathscr{C}$ made of real functions on Ω, then for $f \in \mathscr{C}_{sa}$ one writes $f \geq 0$ if and only if $f(x) \geq 0$ for any $x \in \Omega$. In addition, any $f \geq 0$ has a unique positive square root in $\mathscr{C}$, namely the function $x \mapsto \sqrt{f(x)}$. This notion of positivity endowed $\mathscr{C}_{sa}$ with a partial order: if $f, g \in \mathscr{C}_{sa}$ one sets $f \geq g$ if and only if $f - g \geq 0$. We shall now define a similar partial order on an arbitrary C^*-algebra.

Let $\mathscr{C}$ be a C^*-algebra, and $A \in \mathscr{C}$. One says that A is *positive* if A is self-adjoint, and $\sigma(A) \subset [0, \infty)$. We also write $A \geq 0$ to mean that A is positive, and denote by $\mathscr{C}^+$ the set of positive elements in $\mathscr{C}$. If $\mathscr{J}$ is a subalgebra of $\mathscr{C}$, one clearly has $\mathscr{J}^+ = \mathscr{J} \cap \mathscr{C}^+$.

Theorem 2.5.1. *Let $\mathscr{C}$ be a C^*-algebra and let $A \in \mathscr{C}^+$. Then there exists a unique $B \in \mathscr{C}^+$ such that $B^2 = A$.*

Proof. That there exists $B \in C^*(A)$ such that $B \geq 0$ and $B^2 = A$ follows from the Gelfand representation, since we may use it and identify $C^*(A)$ with $C_0(\Omega)$, where $\Omega := \Omega\big(C^*(A)\big)$, and then apply the above observation, see also Proposition 2.4.17.

Now, suppose that there exists another element $C \in \mathscr{C}^+$ such that $C^2 = A$. Since C commute with A, C also commute with the elements generated by A, and therefore C commute with B. So, let us set $\mathscr{Q} := C^*(\{B, C\})$ which is an abelian C^*-subalgebra of $\mathscr{C}$, and let $\varphi : \mathscr{Q} \to C_0\big(\Omega(\mathscr{Q})\big)$ be its Gelfand representation. Then, $\varphi(C)$ and $\varphi(B)$ are positive square root of $\varphi(A)$, which means that $\varphi(C) = \varphi(B)$. Since φ is is an isometric $*$-isomorphism, it follows that $C = B$. □

If A is a positive element of a C^*-algebra $\mathscr{C}$, we usually write $A^{1/2}$ for its unique positive square root in $\mathscr{C}$. For $A, B \in \mathscr{C}_{sa}$ we also set $A \geq B$ if $A - B \geq 0$. Let us add some elementary information about $\mathscr{C}^+$

Proposition 2.5.2. *Let $\mathscr{C}$ be a C^*-algebra. Then,*

(i) The sum of two positive elements of $\mathscr{C}$ is a positive element of $\mathscr{C}$,

(ii) The set $\mathscr{C}^+$ is equal to $\{A^*A \mid A \in \mathscr{C}\}$,

(iii) *If* $A, B \in \mathscr{C}_{as}$ *and* $C \in \mathscr{C}$, *then* $A \geq B \Rightarrow C^*AC \geq C^*BC$,

(iv) *If* $A \geq B \geq 0$, *then* $A^{1/2} \geq B^{1/2}$,

(v) *If* $A \geq B \geq 0$, *then* $\|A\| \geq \|B\|$,

(vi) *If* $\mathscr{C}$ *is unital and* A, B *are positive and invertible elements of* $\mathscr{C}$, *then* $A \geq B \Rightarrow B^{-1} \geq A^{-1} \geq 0$,

(vii) *For any* $A \in \mathscr{C}$ *there exist* $A_1, A_2, A_3, A_4 \in \mathscr{C}^+$ *such that*

$$A = A_1 - A_2 + iA_3 - iA_4.$$

Proof. See Lemma 2.2.3, Theorem 2.2.5 and Theorem 2.2.6 of [Mur90]. □

Let us stress that the implication $A \geq B \geq 0 \Rightarrow A^2 \geq B^2$ is NOT true in general.

Definition 2.5.3. *For a* C^**-algebra* $\mathscr{C}$, an approximate unit *is an upwards-directed set* $\{I_j\}_{j\in J} \subset \mathscr{C}^+$ *with* $\|I_j\| \leq 1$ *and such that* $A = \lim_j I_j A$ *for any* $A \in \mathscr{C}$.

In order to show that each C^*-algebra $\mathscr{C}$ possesses such an approximate unit, let us first observe that the set of elements of $\mathscr{C}^+$ with norm strictly less than 1 is a partially ordered set which is upwards-directed ($\Leftrightarrow$ if $A, B \in \mathscr{C}^+$ then there exists $C \in \mathscr{C}^+$ such that $C \geq A$ and $C \geq B$). For that purpose, let us set $\mathscr{C}_1^+ := \{A \in \mathscr{C}^+ \mid \|A\| < 1\}$. Observe first that if $A \in \mathscr{C}^+$, then $\mathbf{1} + A$ is invertible in $\tilde{\mathscr{C}}$, and $A(\mathbf{1} + A)^{-1} = \mathbf{1} - (\mathbf{1} + A)^{-1} \in \mathscr{C}$. We next show that if $A, B \in \mathscr{C}^+$ with $B \geq A$, then $B(\mathbf{1} + B)^{-1} \geq A(\mathbf{1} + A)^{-1}$. Indeed, if $B \geq A \geq 0$, then $\mathbf{1} + B \geq \mathbf{1} + A$ in $\tilde{\mathscr{C}}$, and by Proposition 2.5.2.(vi) it follows that $(\mathbf{1} + A)^{-1} \geq (\mathbf{1} + B)^{-1}$. As a consequence, $\mathbf{1}-(\mathbf{1}+B)^{-1} \geq \mathbf{1}-(\mathbf{1}+A)^{-1}$, that is $B(\mathbf{1}+B)^{-1} \geq A(\mathbf{1}+A)^{-1}$ in $\mathscr{C}$. Observe now that if $A \in \mathscr{C}^+$, then $A(\mathbf{1}+A)^{-1} \in \mathscr{C}_1^+$ (use the Gelfand representation applied to $C^*(\{A, \mathbf{1}\})$). Suppose finally that $A, B \in \mathscr{C}_1^+$, and set $A' := A(\mathbf{1} - A)^{-1}$, $B' := B(\mathbf{1} - B)^{-1}$ and $C := (A' + B')(\mathbf{1} + A' + B')^{-1}$. Then, $C \in \mathscr{C}_1^+$, and since $A' + B' \geq A'$ we have $C \geq A'(\mathbf{1} + A')^{-1} = A$. Similarly, $C \geq B$, and therefore $\mathscr{C}_1^+$ is upwards-directed, as claimed.

Theorem 2.5.4. *Every* C^**-algebra* $\mathscr{C}$ *admits an approximate unit.*

The idea of the proof is to show that the upwards-directed set $\mathscr{C}_1^+$ provide such an approximate unit. More precisely, for any $\Lambda \in \mathscr{C}_1^+$, we set $I_\Lambda := \Lambda$ and show that the family $\{I_\Lambda\}_{\Lambda\in\mathscr{C}_1^+}$ is an approximate unit. This approximate unit is called *the canonical approximate unit*. We refer to [Mur90, Thm. 3.1.1] for the details. Note that in the applications, more natural approximate units appear quite often.

If $\{I_j\}_{j\in J}$ is an approximate unit for a C^*-algebra, then, one has by definition $\lim_j \|(\mathbf{1} - I_j)A\| = 0$ for all $A \in \mathscr{C}$. Let us also observe that $\lim_j \|A(\mathbf{1} - I_j)\| = 0$. Indeed, from the relations

$$\|A(\mathbf{1} - I_j)\|^2 = \|(\mathbf{1} - I_j)A^*A(\mathbf{1} - I_j)\| \leq \|(\mathbf{1} - I_j)A^*A\|$$

one directly infers the statement.

Theorem 2.5.5. *Let $\mathscr{J}$ be a closed self-adjoint ideal in a C^*-algebra $\mathscr{C}$. Since $\mathscr{J}$ is itself a C^*-algebra, there exists an approximate unit $\{I_j\}_{j\in J}$ for $\mathscr{J}$, and then for each $A \in \mathscr{C}$ one has*

$$\|A + \mathscr{J}\| = \lim_j \|A - I_j A\| = \lim_j \|A - AI_j\|$$

Proof. Let $A \in \mathscr{C}$ and let $\varepsilon > 0$. From the definition of the norm of $A + \mathscr{J}$ there exists $B \in \mathscr{J}$ such that $\|A + B\| < \|A + \mathscr{J}\| + \varepsilon/2$. Since $B = \lim_j I_j B$ there exists j_0 such that $\|(\mathbf{1} - I_i)B\| < \varepsilon/2$ for all $j \geq j_0$, and therefore

$$\begin{aligned}\|A - I_j A\| &\leq \|(\mathbf{1} - I_j)(A + B)\| + \|(\mathbf{1} - I_j)B\| \leq \|A + B\| + \|(\mathbf{1} - I_j)B\| \\ &< \|A + \mathscr{J}\| + \varepsilon.\end{aligned}$$

It follows that $\|A + \mathscr{J}\| = \lim_j \|A - I_j A\|$. The second equality can be shown similarly. □

Let us now state three useful corollaries which can be deduced from this statement, and refer to [Mur90, Sec. 3.1] for their proofs. These statements correspond to extensions to the framework of C^*-algebras of results which have already been discussed for Banach algebras.

Corollary 2.5.6. *If $\mathscr{J}$ is a closed self-adjoint ideal in a C^*-algebra, then the quotient algebra $\mathscr{C}/\mathscr{J}$ is a C^*-algebra.*

Corollary 2.5.7. *If $\varphi : \mathscr{C} \to \mathscr{Q}$ is an injective $*$-homomorphism between C^*-algebras, then φ is necessarily isometric.*

Corollary 2.5.8. *If $\varphi : \mathscr{C} \to \mathscr{Q}$ is a $*$-homomorphism between C^*-algebras, then $\varphi(\mathscr{C})$ is a C^*-subalgebra of $\mathscr{Q}$.*

Extension 2.5.9. *With the use of an approximate unit, give the proof the three corollaries.*

We now state an important result for the theory of C^*-algebra, the GNS construction. It will then allow us to consider any C^*-algebra as a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

Definition 2.5.10. A representation *of a C^*-algebra $\mathscr{C}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and $\pi : \mathscr{C} \to \mathscr{B}(\mathcal{H})$ is a $*$-homomorphism. This representation is* faithful *if π is injective.*

Theorem 2.5.11 (Gelfand-Naimark-Segal (GNS) representation)**.** *For any C^*-algebra $\mathscr{C}$ there exists a faithful representation.*

Extension 2.5.12. *The proof of this theorem is based on the notion of* states *(positive linear functionals) on a C^*-algebra, and on the existence of sufficiently many such states. The construction is rather explicit and can be studied.*

With the GNS construction at hand, we can end this chapter by considering again the multiplier algebra $\mathscr{M}(\mathscr{C})$ for a C^*-algebra $\mathscr{C}$, and add some information concerning this algebra. More precisely, let us assume that the C^*-algebra $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ acts non-degenerately on $\mathcal{H}$, *i.e.* for any $f \in \mathcal{H} \setminus \{0\}$ there exists $A \in \mathscr{C}$ such that $Af \neq 0$. Note that this is not really any constraint since one can always "eliminate" any superfluous part of the Hilbert space. Then it is natural to set

$$\mathscr{M}_{\mathcal{H}}(\mathscr{C}) := \big\{B \in \mathscr{B}(\mathcal{H}) \mid BA \in \mathscr{C} \text{ and } AB \in \mathscr{C} \text{ for all } A \in \mathscr{C}\big\}.$$

Theorem 2.5.13. *Let $\mathscr{C}$ be a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ acting non-degenerately on $\mathcal{H}$. Then, the correspondence*

$$\mathscr{M}_{\mathcal{H}}(\mathscr{C}) \ni C \mapsto (L_C, R_C) \in \mathscr{M}(\mathscr{C})$$

is an isometric $*$-isomorphism.

We refer to [W-O93, Prop. 2.2.11] for the proof of this statement. Note that the non-trivial part of the proof consists in constructing the inverse map $\mathscr{M}(\mathscr{C}) \to \mathscr{M}_{\mathcal{H}}(\mathscr{C})$. Because of the previous results, we shall simply write $\mathscr{M}(\mathscr{C})$ for $\mathscr{M}_H(\mathscr{C})$ and also call it *the multiplier algebra*. This should not lead to any confusion.

Definition 2.5.14. *Let $\mathscr{C} \subset \mathscr{B}(\mathcal{H})$ be a C^*-algebra acting non-degenerately on $\mathcal{H}$.* The strict topology *on $\mathscr{M}(\mathscr{C})$ is the weakest topology making the maps $B \mapsto BA$ and $B \mapsto AB$ norm continuous, for any $B \in \mathscr{M}(\mathscr{C})$ and $A \in \mathscr{C}$. In other words, the strict topology is the topology generated by the family of seminorms $B \mapsto \|BA\|$ and $B \mapsto \|AB\|$.*

It can be shown that $\mathscr{M}(\mathscr{C})$ is strictly complete, or equivalently that every strict Cauchy net in $\mathscr{M}(\mathscr{C})$ is strictly convergent in $\mathscr{M}(\mathscr{C})$. In fact, $\mathscr{M}(\mathscr{C})$ is the strict completion of $\mathscr{C}$. We refer to Section 2.3 of [W-O93] for a friendly approach to the strict topology.

Chapter 3

Crossed product C^*-algebras

The aim of this chapter is to present an introduction to the theory of crossed product C^*-algebras. Our main references will be Chapter 7 of [Ped79] and the first chapters of [Wil07]. We shall also heavily rely on [Fol95] for the preliminary sections on locally compact groups, and refer to [Tom87] for a nice introduction to C^*-dynamical systems and crossed product algebras in a restricted setting.

3.1 Locally compact groups

We start with some information on locally compact group. The main reference is [Fol95].

Definition 3.1.1. *A locally compact group is a group G equipped with a locally compact and Hausdorff*[1] *topology with respect to which the group operations are continuous,* i.e. *$G \times G \ni (x, y) \mapsto xy \in G$ is continuous, and $G \ni x \mapsto x^{-1} \in G$ is continuous. The unit of G is denoted by* 1*.*

Note that we use the multiplicative notation for the group, and therefore the unit is denoted by 1. If the additive notation is used (and this will be the case at some places in the sequel), then the continuity requirements read $G \times G \ni (x, y) \mapsto x + y \in G$ is continuous, and $G \ni x \mapsto -x \in G$ is continuous, and the unit of G is denoted by 0. In the sequel, G will always denote a locally compact group.

If V is a subset of G, we write $V^{-1} := \{x^{-1} \in G \mid x \in V\}$ and say that V is *symmetric* if $V = V^{-1}$. For two subsets V_1, V_2 of G, we write V_1V_2 for $\{xy \in G \mid x \in V_1 \text{ and } y \in V_2\}$. A subgroup of H of G is *normal* if $xHx^{-1} = H$ for all $x \in G$. In particular, if H is a normal subgroup of G, then its quotient G/H is also a locally compact group.

For any bounded map $f : G \to \mathbb{C}$, we define the left and right translates of f by $[L_y f](x) := f(y^{-1}x)$ and $[R_y f](x) := f(xy)$. These definitions make the maps $y \mapsto L_y$ and $y \mapsto R_y$ group homomorphisms. We say that f is *left uniformly continuous*,

[1] This condition is often tacitly assumed in the literature, and we will always assume it implicitly.

resp. *right uniformly continuous*, if $\|L_y f - f\|_\infty \to 0$, resp. $\|R_y f - f\|_\infty \to 0$, as $y \to 1$ in G.

Let us start with a simple result which is well-known for continuous functions on $\mathbb{R}^d$ with compact support. We use the notation $C_c(G)$ for the set of compactly supported continuous complex functions on G.

Lemma 3.1.2. *If $f \in C_c(G)$, then f is left and right uniformly continuous.*

Proof. We give the proof for the right uniform continuity, the argument for the other one is similar. Let $f \in C_c(G)$, $K := \operatorname{supp} f$ and $\varepsilon > 0$. For every $x \in K$, let U_x be a neighbourhood of 1 such that $|f(xy) - f(x)| < \varepsilon/2$ for any $y \in U_x$, and let V_x be a symmetric neighbourhood of 1 such that $V_x V_x \subset U_x$. The sets xV_x define a covering of K, so there exists $x_1, \dots, x_n \in K$ such that $K \subset \cup_{j=1}^n x_j V_{x_j}$. Let us set $V := \cap_{j=1}^n V_{x_j}$ and show that $\sup_{y \in V} \|R_y f - f\|_\infty < \varepsilon$. Indeed, for any $x \in K$ there exists $j \in \{1, \dots, n\}$ such that $x_j^{-1} x \in V_{x_j}$, so that $xy = x_j(x_j^{-1}x)y \in x_j U_{x_j}$ for any $y \in V$. Then, one has

$$|f(xy) - f(x)| \leq |f(xy) - f(x_j)| + |f(x_j) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Similarly, if $xy \in K$, one obtains $|f(xy) - f(x)| < \varepsilon$. If neither $xy \in K$ nor $x \in K$, then $f(xy) - f(x) = 0 - 0 = 0$, and the statement is proved. □

Definition 3.1.3. A left Haar measure*, resp.* a right Haar measure*, on G is a non-zero Radon measure*[2] *μ on G that satisfies $\mu(xV) = \mu(V)$, resp. $\mu(Vx) = \mu(V)$, for every Borel set $V \subset G$ and every $x \in G$.*

For any Radon measure μ and any set V we write $(\tilde{\mu})(V) := \mu(V^{-1})$. From now on, we also denote by $C_c^+(G)$ the subset of compactly supported continuous functions on G which are non-negative.

Lemma 3.1.4. *Let μ be a Radon measure on G.*

(i) μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure,

(ii) μ is a left Haar measure if and only if $\int_G L_y f \,\mathrm{d}\mu = \int_G f \,\mathrm{d}\mu$ for any $f \in C_c^+(G)$ and any $y \in G$.

The proof of this statement is rather simple, see [Fol95, Prop. 2.9]. In view of this result, it is not really relevant to consider differently a left Haar measure or a right Haar measure. We shall follow the more common choice which consists in considering left Haar measures only.

The following statement is of fundamental importance for performing analysis on locally compact groups. We refer to Theorem 2.10 and 2.20 of [Fol95] for its proof, and for various examples of locally compact group with their Haar measure.

[2] A Radon measure is a measure on the algebra of Borel sets of a Hausdorff topological space X that is locally finite and inner regular.

Theorem 3.1.5. *Every locally compact group possesses a left Haar measure, which is unique up to a scaling constant.*

Note that if μ is a Haar measure on G, then $\mu(V) > 0$ for every non-empty open set V, and that $\int_G f\mathrm{d}\mu > 0$ for any $f \in C_c^+(G)$ with $f \neq 0$.

Extension 3.1.6. *It has not been assumed that the locally compact group G is σ-compact ($\Leftrightarrow$ the union of countably many compact subsets). Consequently, the Haar measure is not always σ-finite ($\Leftrightarrow$ G might not a countable union of measurable sets with finite measure). In such a situation, some standard results of analysis which are well-known on $\mathbb{R}^d$ present some complications for their generalization on G, but these problems are manageable, see [Fol95, Sec. 2.3] for details.*

Let us fix a locally compact group G with a left Haar measure μ. We shall denote by $L^p(G, \mathrm{d}\mu)$ the L^p-spaces constructed with this measure. Now, for any $x \in G$ and $V \subset G$, let us define the measure μ_x by $\mu_x(V) := \mu(Vx)$. μ_x is again a left Haar measure, and by Theorem 3.1.5 there exists $\Delta(x) \in (0, \infty)$ such that $\mu_x = \Delta(x)\mu$. Note that the value $\Delta(x)$ is independent of the original choice for the Haar measure μ. The map $\Delta : G \to \mathbb{R}_+$ is called *the modular function of* G. An important result concerning this function is:

Lemma 3.1.7. *The map Δ is a continuous homomorphism from G to the group multiplicative on $\mathbb{R}_+$. Moreover, for any $f \in L^1(G, \mathrm{d}\mu)$ one has $\int_G R_y f \mathrm{d}\mu = \Delta(y^{-1}) \int_G f \mathrm{d}\mu$.*

Proof. For any $x, y \in G$ and $V \subset G$ one has

$$\Delta(xy)\mu(V) = \mu(Vxy) = \Delta(y)\mu(Vx) = \Delta(y)\Delta(x)\mu(V),$$

so that Δ is a homomorphism from G to $\mathbb{R}_+$. For the rest of the proof, we refer to [Fol95, Prop. 2.24]. □

Definition 3.1.8. *A locally compact group G is called* unimodular *if* $\Delta = 1$.

Abelian groups and discrete groups are unimodular, but many others groups are unimodular too.

Lemma 3.1.9. *If K is any compact subgroup of G, then $\Delta|_K = 1$.*

Proof. $\Delta(K)$ is a compact subgroup of $\mathbb{R}_+$, and therefore $\Delta(K) = \{1\}$. □

Corollary 3.1.10. *If G is compact, then G is unimodular.*

Let us now denote by $M(G)$ the space of all complex bounded Radon measures on G, and endow this set with a convolution and an involution: For any $\mu, \nu \in M(G)$ and $f \in C_0(G)$ we define the convolution $\mu * \nu$ by the formula

$$\int_G f(x)\mathrm{d}(\mu * \nu)(x) = \int_G \int_G f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$$

and the involution by the formula $\int_G f(x)\mathrm{d}\mu^*(x) = \int_G \overline{f(x^{-1})\mathrm{d}\mu(x)}$. Endowed with this product and involution, $M(G)$ becomes a B^*-algebra.

The closed self-adjoint ideal $L^1(G)$ consisting of measures which are absolutely continuous with respect to a left Haar measure on G can clearly be identified with the space $L^1(G, \mathrm{d}\mu)$. With this identification one has for any $f, g \in L^1(G)$

$$f * g(x) = \int_G f(y)g(y^{-1}x)\mathrm{d}\mu(y) = \int_G f(xy)g(y^{-1})\mathrm{d}\mu(y)$$

and $f^*(x) = \Delta(x)^{-1}\overline{f(x^{-1})}$. The B^*-algebra $L^1(G)$ is called *the L^1-group algebra of G*. Note that $M(G)$ is always unital, with unit δ_1 (the point mass at 1) while $L^1(G)$ is unital if and only if G is discrete. However, approximate unit exists for $L^1(G)$:

Theorem 3.1.11. *For any locally compact group G, there exists an approximate unit for $L^1(G)$,* i.e. *there exists an increasing net $\{I_j\}_{j\in J} \subset L^1(G)$ with $I_j \geq 0$, $I_j(x^{-1}) = I_j(x)$, and $\int_G I_j(x)\mathrm{d}\mu(x) = 1$, such that* $\lim_j \|f * I_j - f\|_1 = 0$.

Proof. We refer to [Fol95, Prop. 2.42] for a constructive proof. □

Exercise 3.1.12. *We state in this exercise a couple of useful formulas which can be deduced from the definition of the modular function. Let $f \in C_c(G)$ and $x \in G$:*

$$\begin{aligned}
\int_G f(xy)\,\mathrm{d}\mu(y) &= \int_G f(y)\,\mathrm{d}\mu(y), \\
\int_G f(yx)\,\mathrm{d}\mu(y) &= \Delta(x)^{-1}\int_G f(y)\,\mathrm{d}\mu(y), \\
\int_G \Delta(y^{-1})\,f(y^{-1})\,\mathrm{d}\mu(y) &= \int_G f(y)\,\mathrm{d}\mu(y).
\end{aligned}$$

We end this section with some information on representations. In particular, we shall prove a result about the equivalence between unitary representations of the group and non-degenerate representations of the corresponding L^1-group algebra. Note that we use the notation $\mathscr{U}(\mathcal{H})$ for the set of all unitary operators in a Hilbert space $\mathcal{H}$.

Definition 3.1.13. A unitary representation *of G is a pair $(\mathcal{H}, U)$, where $\mathcal{H}$ is a Hilbert space and where $U : G \to \mathscr{U}(\mathcal{H})$ is a homomorphism which is strongly continuous. One usually writes U_x for $U(x) \in \mathscr{U}(\mathcal{H})$.*

Note that on $\mathscr{U}(\mathcal{H})$, weak and strong topologies coincide. Recall also that a representation of a B^*-algebra $\mathscr{C}$ is a pair $(\mathcal{H}, \pi)$ with $\mathcal{H}$ a Hilbert space and $\pi : \mathscr{C} \to \mathscr{B}(\mathcal{H})$ a continuous $*$-homomorphism. This representation is non-degenerate if for any $h \in \mathcal{H}$ there exists $A \in \mathscr{C}$ such that $\pi(A)h \neq 0$.

Proposition 3.1.14. *There are bijective correspondences between the sets of unitary representations of G, representations of $M(G)$ whose restrictions to $L^1(G)$ are non-degenerate, and non-degenerate representations of $L^1(G)$.*

Proof. If $(\mathcal{H}, U)$ is a unitary representation of G, we define for each $\mu \in M(G)$ and each $h, h' \in \mathcal{H}$

$$\langle \pi(\mu)h, h' \rangle := \int_G \langle U_x h, h' \rangle \mathrm{d}\mu(x). \tag{3.1.1}$$

Then $(\mathcal{H}, \pi)$ us a representation of $M(G)$, and by using an approximate unit for $L^1(G)$ we can check that the restriction to $L^1(G)$ is non-degenerate.

Conversely, let $(\mathcal{H}, \pi)$ be a non-degenerate representation of $L^1(G)$ and let $\{I_j\}_{j \in J}$ be an approximate unit for $L^1(G)$. Since elements of the form $\pi(f)h$ with $f \in L^1(G)$ and $h \in \mathcal{H}$ are dense in $\mathcal{H}$, it follows that $\{\pi(I_j)\}$ converges strongly to the operator $\mathbf{1}$. In addition, this representation can be extended to a representation of $L(G)$ (we use the same symbols for this extension) by defining

$$\pi(\mu)(\pi(f)h) = \pi(\mu * f)h \tag{3.1.2}$$

for any $\mu \in M(G)$, $f \in L^1(G)$ and $h \in \mathcal{H}$. Equivalently, one has

$$\pi(\mu)h = s - \lim_j \pi(\mu * I_j)h, \tag{3.1.3}$$

which shows that the extension is unique. The restriction of $M(G)$ to point measures δ_x with $x \in G$, provides then a unitary representation of G whose extension to $L^1(G)$ is precisely the representation $(\mathcal{H}, \pi)$. □

We refer to [Fol95, Sec. 3.2] for more details in the above proof. Note that a unitary representation of G always exists, namely its *left regular representation*: We consider $\mathcal{H} := L^2(G, \mathrm{d}\mu)$ where μ is a Haar measure on G, and set

$$[U_x f](y) = [L_x f](y) = f(x^{-1}y). \tag{3.1.4}$$

By the construction exhibited in the proof of the previous proposition, one also obtains a non-degenerate representation of $L^1(G)$ on $L^2(G, \mathrm{d}\mu)$, whose norm closure in $\mathscr{B}\big(L^2(G, \mathrm{d}\mu)\big)$ is called *the reduced group C^*-algebra*, and is usually denoted by $\mathscr{C}^*_r(G)$. On the other hand, the completion of $L^1(G)$ with the norm

$$\|f\| := \sup\{\|\pi(f)\| \mid (\mathcal{H}, \pi) \text{ is a unitary representation of } G\}$$

is called *the group C^*-algebra $\mathscr{C}^*(G)$*.

Let us now consider a unitary representation $(\mathcal{H}, U)$ of G. If there exists a non-trivial closed subspace $\mathcal{M}$ of $\mathcal{H}$ such that $U_x\mathcal{M} \subset \mathcal{M}$ for all $x \in G$, then $\mathcal{M}$ is called *an invariant subspace for U*. In such a case, the restriction $(\mathcal{M}, U|_{\mathcal{M}})$ is a unitary *subrepresentation* of G. If such a subrepresentation exists, the original representation $(\mathcal{H}, U)$ is called *reducible*, and otherwise *irreducible*.

The following statement is important in this context. Its proof is not difficult but requires some preliminary lemmas, see [Fol95, Lem. 3.5].

Theorem 3.1.15 (Schur's Lemma)**.** *A unitary representation $(\mathcal{H}, U)$ of G is irreducible if and only if the set of elements of $\mathscr{B}(\mathcal{H})$ which commute with U_x for all $x \in G$ is reduced to $\mathbb{C}\mathbf{1}$.*

The set mentioned in the previous statement is usually called *the commutant* or *the centralizer* of $(\mathcal{H}, U)$.

Corollary 3.1.16. *If G is abelian, then every irreducible representation of G is one-dimensional.*

Proof. If $(\mathcal{H}, U)$ is a representation of G, then U_x commute will all elements U_y for any $y \in G$. Therefore, U_x belongs to the commutant of $(\mathcal{H}, U)$ for any $x \in G$. If this representation is irreducible, this commutant is equal to $\mathbb{C}\mathbf{1}$, and therefore we have $U_x = c_x\mathbf{1}$, with $c_x \in \mathbb{C}$, for all $x \in G$. Since every one-dimensional subspace of $\mathcal{H}$ is then invariant for U, it follows that $\dim(\mathcal{H}) = 1$. □

Extension 3.1.17. *The notion of* amenable *locally compact group is important and could be studied,* cf. *[Ped79, Sec. 7.3]. Note that abelian groups and compact groups are amenable.*

3.2 Locally compact abelian groups

We shall now develop the theory of locally compact abelian groups, and refer to [Fol95, Sec. 4] for more details. In particular, one of our aims is to show that the usual Fourier transform is nothing but a Gelfand representation in the context of locally compact abelian groups.

In the section, G will always denote a locally compact abelian group. For them, left and right continuity coincide, convolution is commutative, and the modular function is identically equal to 1. For simplicity, we shall simply denote by $\mathrm{d}x$ a Haar measure on G (which is unique up to a scaling constant), and $L^p(G)$ for $L^p(G, \mathrm{d}x)$ with the norm denoted by $\|\cdot\|_p$.

Let us recall from Corollary 3.1.16 that all unitary irreducible representation of G are one-dimensional. Thus, for each such representation $(\mathcal{H}, U)$ one can take $\mathcal{H} = \mathbb{C}$ and then $U_x = \xi(x)$, where $\xi : G \to \mathbb{T}$ is a continuous homomorphism.

Definition 3.2.1. *For a locally compact abelian group G,* a character *ξ is a continuous homomorphism from G to $\mathbb{T}$. The set of all characters is denoted by $\hat{G}$.*

Note that we shall use both notations $\xi(x)$ or $\langle x, \xi\rangle$. As a consequence of Proposition 3.1.14, this unitary representation induces a non-degenerate representation $(\mathbb{C}, \tau_\xi)$ of $L^1(G)$ by the formula

$$\tau_\xi(f) = \int_G \langle x, \xi\rangle f(x)\mathrm{d}x \tag{3.2.1}$$

for any $f \in L^1(G)$. Since $\mathscr{B}(\mathbb{C})$ is clearly identified with $\mathbb{C}$, such a representation is nothing but a character on the algebra $L^1(G)$, *i.e.* an element of $\Omega\big(L^1(G)\big)$, see Definition 2.3.1. Conversely, any character τ on $L^1(G)$ defines a character on G. Indeed, observe first that any $\tau \in \Omega\big(L^1(G)\big) = L^1(G)^*$ is obtained by integration against some $\xi \in L^\infty(G)$. Then, choose $f \in L^1(G)$ such that $\tau(f) \neq 0$. For any $g \in L^1(G)$ one has

$$\begin{aligned}\tau(f)\int_G \xi(y)g(y)\mathrm{d}y &= \tau(f)\tau(g) = \tau(f*g)\\ &= \int_G\int_G \xi(x)f(xy^{-1})g(y)\mathrm{d}y\,\mathrm{d}x = \int_G \tau(L_y f)g(y)\mathrm{d}y\end{aligned}$$

so that $\xi(y) = \frac{\tau(L_y f)}{\tau(f)}$ locally a.e. We can thus redefine ξ such that $\xi(y) = \frac{\tau(L_y f)}{\tau(f)}$ for every $y \in G$, and then ξ is continuous. As a consequence, one has

$$\xi(xy)\tau(f) = \tau(L_{xy}f) = \tau(L_x L_y f) = \xi(x)\xi(y)\tau(f)$$

which means $\xi(xy) = \xi(x)\xi(y)$. Finally, $\xi(x^n) = \xi(x)^n$ for any $n \in \mathbb{Z}$, and since ξ is bounded it implies that $|\xi(x)| = 1$. As a consequence, ξ is a character on G, as expected.

We have thus proved that:

Theorem 3.2.2. *For any locally compact abelian group, the set of characters $\hat{G}$ can be identified with $\Omega\big(L^1(G)\big)$ through formula* (3.2.1).

$\hat{G}$ is an abelian group under pointwise multiplication, its identity is the constant function 1 on G, and one has

$$\langle x, \xi^{-1}\rangle = \overline{\langle x, \xi\rangle} = \langle x^{-1}, \xi\rangle.$$

By endowing $\hat{G}$ with the weak* topology inherited from $L^\infty(G)$, one infers that $\hat{G}$ is a locally compact abelian group, called *the dual group of* G. Note that this topology coincides with the one borrowed from $\Omega\big(L^1(G)\big)$ through the identification mentioned above.

Examples 3.2.3. *(i) For $G = \mathbb{R}$, $\hat{G} \cong \mathbb{R}$ with the pairing $\langle x, \xi\rangle = e^{i\xi x}$,*

(ii) For $G = \mathbb{T}$, $\hat{G} \cong \mathbb{Z}$ with the pairing $\langle \alpha, n\rangle = \alpha^n$,

(iii) For $G = \mathbb{Z}$, $\hat{G} \cong \mathbb{T}$ with the pairing $\langle n, \alpha\rangle = \alpha^n$.

Let us add some information in the case of compact or discrete groups.

Lemma 3.2.4. *If G is a compact abelian group with a Haar measure normalized such that $\int_G \mathrm{d}x = 1$, then $\hat{G}$ is an orthonormal set in $L^2(G)$.*

Proof. If $\xi \in \hat{G}$ then $|\xi| = 1$ and therefore $\|\xi\|_2 = 1$. If $\xi, \eta \in \hat{G}$ with $\xi \neq \eta$ there exists $x_0 \in G$ such that $\langle x_0, \xi\eta^{-1}\rangle \neq 1$, and then we have

$$\int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x = \langle x_0, \xi\eta^{-1}\rangle \int_G \langle x_0^{-1}x, \xi\eta^{-1}\rangle \mathrm{d}x = \langle x_0, \xi\eta^{-1}\rangle \int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x,$$

which implies that $\int_G \langle x, \xi\eta^{-1}\rangle \mathrm{d}x = 0$. □

Proposition 3.2.5. *If G is discrete, then $\hat{G}$ is compact. If G is compact, then $\hat{G}$ is discrete.*

Proof. If G is discrete, then $L^1(G)$ has a unit, and therefore $\Omega\big(L^1(G)\big)$ is compact. By Theorem 3.2.2, it follows that $\hat{G}$ is compact.

If G is compact (with Haar measure satisfying $\int_G \mathrm{d}x = 1$), then the constant function 1 belongs to $L^1(G)$. It follows from Lemma 3.2.4 that $\int_G \langle x, \xi\rangle \mathrm{d}x = 1$ if $\xi = 1$ while $\int_G \langle x, \xi\rangle \mathrm{d}x = \langle 1, \xi\rangle_{L^2(G)} = 0$ if $\xi \in \hat{G}$ with $\xi \neq 1$. Since the set $\{f \in L^\infty(G) \mid |\int_G f(x)\mathrm{d}x| > 1/2\}$ is a weak* open set, it follows that $\{1\}$ is an open set in $\hat{G}$, and therefore $\hat{G}$ is discrete. □

Henceforth, it is more convenient (and more common) to use a slightly different identification of $\hat{G}$ with $\Omega\big(L^1(G)\big)$ than the one given in (3.2.1). Namely, we associate with $\xi \in \hat{G}$ the functional

$$f \mapsto \int_G \overline{\langle x, \xi\rangle} f(x)\mathrm{d}x.$$

The Gelfand transform for the abelian Banach algebra $L^1(G)$ becomes then the map $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ defined by

$$[\mathcal{F}f](\xi) \equiv \hat{f}(\xi) = \int_G \overline{\langle x, \xi\rangle} f(x)\mathrm{d}x$$

and is usually called in this context *the Fourier transform.* A rephrasing of Theorem 2.3.5 together with some simple verifications lead to:

Theorem 3.2.6. *The Fourier transform is a norm decreasing $*$-homomorphism from $L^1(G)$ to $C_0(\hat{G})$, or to $C(\hat{G})$ if $\hat{G}$ is compact. It extends to a $*$-isomorphism between $\mathscr{C}^*(G)$ and $C_0(\hat{G})$.*

Extension 3.2.7. *In the setting presented above, many classical results of Fourier analysis on $\mathbb{R}^d$ can be extended to arbitrary locally compact abelian groups. This subject is nicely presented in Section 4 of [Fol95]. A look at Plancherel Theorem, at some Fourier inversions formula or at Pontrjagin duality theorem is certainly valuable.*

3.3 C^*-dynamical systems

In the sequel, we shall go on with the convention of simply writing $\mathrm{d}x$ for a left Haar measure on G, and denote by $L^p(G)$ the spaces $L^p(G, \mathrm{d}x)$.

Definition 3.3.1. A C^*-dynamical system *consists in a triple $(\mathscr{C}, G, \theta)$, where $\mathscr{C}$ is a C^*-algebra, G is a locally compact group, and θ is a continuous homomorphism from G to $\mathsf{Aut}(\mathscr{C})$, with $\mathsf{Aut}(\mathscr{C})$ the group of $*$-automorphisms of $\mathscr{C}$ equipped with the topology of pointwise convergence.*

Note that the topology on $\mathsf{Aut}(\mathscr{C})$ means that for each $A \in \mathscr{C}$, the map

$$G \ni x \mapsto \theta_x(A) \in \mathscr{C}$$

is continuous.

Example 3.3.2. *Let us present an example which will be important later on. We consider the C^*-algebra $\mathscr{C} := BC_u(\mathbb{R}^d)$, the group $G = \mathbb{R}^d$ (with the additive notation) and the action θ of G on $\mathscr{C}$ by translation,* i.e. $[\theta_x f](y) = f(y - x)$ *for any $f \in \mathscr{C}$ and $x, y \in \mathbb{R}^d$. Almost by definition, the algebra $BC_u(\mathbb{R}^d)$ is the largest algebra of functions on $\mathbb{R}^d$ for which this action is continuous, namely $\|\theta_x f - f\|_\infty \to 0$ as $x \to 0$. Then the triple $(\mathscr{C}, G, \theta)$ is a C^*-dynamical system. Note that any C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is stable under translations would also be suitable for such a dynamical system, as for example $C_0(\mathbb{R}^d)$.*

Exercise 3.3.3. *Let G be a locally compact group, Ω be a locally compact space, and assume that the group G acts continuously on Ω,* i.e. *there exists a continuous map*

$$G \times \Omega \ni (x, \xi) \mapsto x \cdot \xi \in \Omega$$

such that $1 \cdot \xi = \xi$ and $x \cdot (y \cdot \xi) = xy \cdot \xi$ for all $x, y \in G$ and $\xi \in \Omega$. Such a system is called a locally compact transformation group, *and Ω is also called* a locally compact G-space. *Then, let us define an automorphism of $C_0(\Omega)$ by $[\theta_x f](\xi) := f(x^{-1} \cdot \xi)$ for any $f \in C_0(\Omega)$, $x \in G$ and $\xi \in \Omega$. Check that the triple $\big(C_0(\Omega), G, \theta\big)$ is a C^*-dynamical system. In fact, it turns out that all C^*-dynamical systems with $\mathscr{C}$ abelian arise from locally compact transformation groups, see [Wil07, Prop. 2.7] for details.*

Definition 3.3.4. A covariant representation *of a C^*-dynamical system $(\mathscr{C}, G, \theta)$ consists in a triple $(\mathcal{H}, \pi, U)$, where $(\mathcal{H}, \pi)$ is a representation of $\mathscr{C}$, $(\mathcal{H}, U)$ is a unitary representation of G, and the following compatibility condition holds*

$$\pi\big(\theta_x(A)\big) = U_x \pi(A) U_x^*$$

for all $A \in \mathscr{C}$ and $x \in G$. This covariant representation is non-degenerate if the representation $(\mathcal{H}, \pi)$ of $\mathscr{C}$ is non-degenerate.

Examples 3.3.5. *Covariant representations of the dynamical systems $(\mathscr{C}, \{1\}, id)$ correspond exactly to representation of $\mathscr{C}$. On the other hand, covariant representations of the dynamical systems $(\mathbb{C}, G, id)$ coincide with unitary representations of G.*

Example 3.3.6 (Regular representation)**.** *Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let $(\mathcal{H}, \pi)$ be a representation of $\mathscr{C}$. Consider the Hilbert space $L^2(G; \mathcal{H}) \cong L^2(G) \otimes \mathcal{H}$, and let us then define $\tilde{\pi} : \mathscr{C} \to \mathscr{B}\big(L^2(G; \mathcal{H})\big)$ and $\tilde{U} : G \to \mathscr{U}\big(L^2(G; \mathcal{H})\big)$ by*

$$[\tilde{\pi}(A)h](x) := \pi\big(\theta_x^{-1}(A)\big)h(x) \qquad \text{and} \qquad [\tilde{U}_x h](y) := h(x^{-1}y),$$

for any $A \in \mathscr{C}$, $h \in L^2(G;\mathcal{H})$ and $x, y \in G$. Let us now check that

$$\begin{aligned}[\tilde{U}_x \tilde{\pi}(A) \tilde{U}_x^* h](y) = [\tilde{\pi}(A)\tilde{U}_x^* h](x^{-1}y) &= \pi\big(\theta^{-1}_{x^{-1}y}(A)\big)\big(\tilde{U}_x^* h(x^{-1}y)\big) \\ &= \pi\big[\theta_y^{-1}\big(\theta_x(A)\big)\big]\big(h(y)\big) = \big[\tilde{\pi}\big(\theta_x(A)\big)h\big](y).\end{aligned}$$

Thus, the triple $\big(L^2(G;\mathcal{H}), \tilde{\pi}, \tilde{U}\big)$ is a covariant representation of the C^*-dynamical system, called its regular representation. *As a consequence, any C^*-dynamical system has at least one covariant representation. It can also be shown that the regular representation is non-degenerate if the representation $(\mathcal{H},\pi)$ of $\mathscr{C}$ is non-degenerate,* cf. *[Wil07, Lem. 2.17].*

Exercise 3.3.7. *Let G be a locally compact group and its left action on elements of $C_0(G)$,* i.e. *$[L_x f](y) = f(x^{-1}y)$. In this setting, check that $\big(C_0(G), G, L\big)$ is a C^*-dynamical system. Now, let $\mathcal{H} := L^2(G)$ and define $\mathrm{Id} : C_0(G) \to \mathscr{B}(\mathcal{H})$ be the identification map defined by $[\mathrm{Id}(f)h](x) = f(x)h(x)$ for any $f \in C_0(G)$ and $h \in \mathcal{H}$. Finally, let $U_x \in \mathscr{U}(\mathcal{H})$ defined by $[U_x h](y) = h(x^{-1}y)$. Check that $(\mathcal{H}, \mathrm{Id}, U)$ is a covariant representation of $\big(C_0(G), G, L\big)$.*

3.4 Crossed product algebras

This section is mainly based on [Ped79, Sec. 7.6] together with [Wil07, Sec. 2.3]. However, note that quite a lot of explicit computations are explicitly written in [Sko12].

Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let us define a product and an involution on the linear space $C_c(G;\mathscr{C})$ of continuous functions from G to $\mathscr{C}$ with compact support: for any $f, g \in C_c(G;\mathscr{C})$ and $x \in G$ one sets

$$\begin{aligned}[f * g](x) &:= \int_G f(y)\,\theta_y\big(g(y^{-1}x)\big)\mathrm{d}y \\ f^*(x) &:= \Delta(x)^{-1}\theta_x\big(f(x^{-1})^*\big).\end{aligned}$$

Some lengthy but straightforward computations show that these definitions endow $C_c(G;\mathscr{C})$ with an associative product and with an involution. In addition, if one sets $\|f\|_1 := \int_G \|f(y)\|\mathrm{d}y$, then $C_c(G;\mathscr{C})$ becomes a norm algebra with a submultiplicative norm, *i.e.* $\|f * g\|_1 \leq \|f\|_1\|g\|_1$. The completion of $C_c(G;\mathscr{C})$ with this norm is denoted by $L^1(G;\mathscr{C})$ which is therefore a B^*-algebra.

Clearly, if $\mathscr{C} = \mathbb{C}$, the above construction leads simply to the algebra $L^1(G)$. Let us also observe that if $f \in L^1(G)$ and $A \in \mathscr{C}$, then the element $f \otimes A$ is an element of $L^1(G;\mathscr{C})$. In addition, the linear span of elements of the form $f \otimes A$ with $f \in C_c(G)$ and $A \in \mathscr{C}$ is dense in $L^1(G;\mathscr{C})$.

Let us now state an important result relating a covariant representation of a C^*-dynamical system to a representation of the corresponding L^1-algebra:

Theorem 3.4.1. *If* $(\mathcal{H},\pi,U)$ *is a covariant representation of the* C^**-dynamical system* $(\mathscr{C},G,\theta)$*, then there is a norm-decreasing representation* $(\mathcal{H},\pi\rtimes U)$ *of* $L^1(G;\mathscr{C})$ *defined by*

$$\pi\rtimes U(f)=\int_G \pi\big(f(y)\big)U_y\mathrm{d}y \tag{3.4.1}$$

for every $f\in C_c(G;\mathscr{C})$*. Moreover,* $(\mathcal{H},\pi\rtimes U)$ *is non-degenerate if* $(\mathcal{H},\pi)$ *is non-degenerate.*

The representation $(\mathcal{H},\pi\rtimes U)$ is called *the integrated representation of* $(\mathcal{H},\pi,U)$. We provide below a sketch of the proof, and refer to [Wil07, Prop. 2.23] for the details.

Proof. Let $f\in C_c(G;\mathscr{C})$ and define $\pi\rtimes U(f)\in\mathscr{B}(\mathcal{H})$ by (3.4.1). Then, one observes that

$$\begin{aligned}\pi\rtimes U(f^*)&=\int_G \pi\big[\Delta(y)^{-1}\theta_y\big(f(y^{-1})^*\big)\big]U_y\mathrm{d}y\\&=\int_G \Delta(y)^{-1}U_y\pi\big(f(y^{-1})^*\big)\mathrm{d}y=\int_G U_y^*\pi\big(f(y)^*\big)\mathrm{d}y=\big(\pi\rtimes U(f)\big)^*,\end{aligned}$$

and (with $g\in C_c(G;\mathscr{C})$)

$$\begin{aligned}\pi\rtimes U(f*g)&=\int_G\pi\Big[\int_G f(y)\theta_y\big(g(y^{-1}x)\big)\mathrm{d}y\Big]U_x\mathrm{d}x\\&=\int_G\Big[\int_G\pi\big[f(y)\big]U_y\pi\big[g(y^{-1}x)\big]U_y^*U_x\mathrm{d}y\Big]\mathrm{d}x\\&=\int_G\Big[\int_G\pi\big[f(y)\big]U_y\pi\big[g(x)\big]U_x\mathrm{d}y\Big]\mathrm{d}x\\&=\pi\rtimes U(f)\ \pi\rtimes U(g).\end{aligned}$$

In addition, one also has $\|\pi\rtimes U(f)\|\le\int_G\big\|\pi\big(f(y)\big)U_y\big\|\mathrm{d}y=\|f\|_1$. These relations show that $\big(\mathcal{H},\pi\rtimes U\big)$ extends to a norm-decreasing representation of $L^1(G;\mathscr{C})$.

For the non-degeneracy, we refer to the proof of [Wil07, Prop. 2.23]. ☐

Definition 3.4.2. *For any* C^**-dynamical system* $(\mathscr{C},G,\theta)$ *and any* $f\in C_c(G;\mathscr{C})$ *let us set*

$$\|f\|:=\sup\{\|\pi\rtimes U(f)\|_{\mathscr{B}(\mathcal{H})}\mid(\mathcal{H},\pi,U)\text{ is a covariant representation of }(\mathscr{C},G,\theta)\}. \tag{3.4.2}$$

The norm $\|\cdot\|$ *on* $C_c(G;\mathscr{C})$ *is called* the universal norm*, and is dominated by the* $\|\cdot\|_1$*-norm. The completion of* $C_c(G;\mathscr{C})$ *with respect to the norm* $\|\cdot\|$ *is called* the crossed product C^*-algebra of $\mathscr{C}$ by G *and is denoted by* $\mathscr{C}\rtimes_\theta G$*.*

Example 3.4.3. *If* G *is a locally compact group and if* θ *corresponds to the left action on* $C_0(G)$, i.e. $[\theta_x f](y)=f(x^{-1}y)$ *for all* $f\in C_0(G)$*, then* $C_0(G)\rtimes_\theta G$ *is* $*$*-isomorphic to the compact operators on* $L^2(G)$*.*

Remark 3.4.4. *If the C^*-algebra $\mathscr{C}$ is abelian, with $\mathscr{C} \cong C_0(\Omega)$, the corresponding crossed product algebra $\mathscr{C} \rtimes_\theta G$ is also called* transformation group C^*-algebra. *Moreover, it is possible to describe the $*$-algebraic structure on $C_c\big(G; C_0(\Omega)\big)$ in terms of functions on $G \times \Omega$. Indeed, observe first that by obvious identifications one has*

$$C_c(G \times \Omega) \subset C_c\big(G; C_c(\Omega)\big) \subset C_c\big(G; C_0(\Omega)\big).$$

Then, if one denotes the action of G on Ω by $\cdot$ (note that such an action always exists, see Proposition 2.7 of [Wil07]) one ends up with the following formula:

$$[f * g](x,\xi) = \int_G f(y,\xi)\, g(y^{-1}x, y^{-1}\cdot\xi)\mathrm{d}y$$
$$f^*(x,\xi) = \Delta(x)^{-1}\overline{f(x^{-1}, x^{-1}\cdot\xi)}$$

for $f, g \in C_c(X \times \Omega)$ and $(x,\xi) \in G \times \Omega$.

Except in some very special cases, the crossed product algebra $\mathscr{C} \rtimes_\theta G$ contains neither a copy of the algebra $\mathscr{C}$ nor a copy of $L^1(G)$. However, its multiplier algebra $\mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$ does, as we shall observe now. Indeed, for any $A \in \mathscr{M}(\mathscr{C})$, $\mu \in M(G)$ and $f \in C_c(G;\mathscr{C})$ let us define

$$[L_{(A,\mu)}f](x) := A \int_G \theta_y\big(f(y^{-1}x)\big)\,\mathrm{d}\mu(y)$$
$$[R_{(A,\mu)}f](x) := \int_G f(xy^{-1})\theta_{xy^{-1}}(A)\Delta(y)^{-1}\,\mathrm{d}\mu(y).$$

One can check that $L_{(A,\mu)}$ and $R_{(A,\mu)}$ are bounded by $\|A\|\,\|\mu\|$, and thus extend by continuity to linear operators on $L^1(G;\mathscr{C})$. In addition, some straightforward computations (see [Ped79, Lem. 7.6.3]) show that

$$L_{(A,\mu)}(f * g) = (L_{(A,\mu)}f) * g, \qquad R_{(A,\mu)}(f * g) = f * (R_{(A,\mu)}g)$$

and that $(R_{(A,\mu)}f) * g = f * (L_{(A,\mu)}g)$. Thus, the pair $(L_{(A,\mu)}, R_{(A,\mu)})$ defines a double centralizer on the B^*-algebra $L^1(G;\mathscr{C})$, see Section 2.4. With these notions at hand, one can deduce that:

Theorem 3.4.5. *For any C^*-dynamical system $(\mathscr{C}, G, \theta)$, there exist a non-degenerate faithful $*$-homomorphism*

$$i_{\mathscr{C}} : \mathscr{C} \to \mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$$

and an injective homomorphism

$$i_G : G \to \mathscr{M}\big(\mathscr{C} \rtimes_\theta G\big)$$

defined by the formulas $i_{\mathscr{C}}(A) := (L_{(A,\delta_1)}, R_{(A,\delta_1)})$ and $i_G(x) := (L_{(\mathbf{1},\delta_x)}, R_{(\mathbf{1},\delta_x)})$.

Proof. See [Ped79, Sec. 7.6] and Proposition 2.34 of [Wil07] for the details. □

By using the alternative representation of the multiplier algebra, as introduced at the end of Chapter 2, one also infers the following corollary:

Corollary 3.4.6. *Let $(\mathcal{H},\pi,U)$ be a non-degenerate covariant representation of the C^*-dynamical system $(\mathscr{C},G,\theta)$ such that the representation π of $\mathscr{C}$ is faithful. Then, the maps $\mathscr{C}\ni A\mapsto\pi(A)\in\mathscr{B}(\mathcal{H})$ and $G\ni x\mapsto U_x\in\mathscr{U}(\mathcal{H})$ are injective homomorphisms into $\mathscr{M}\big(\pi\rtimes U(\mathscr{C}\rtimes_\theta G)\big)\subset\mathscr{B}(\mathcal{H})$.*

Proof. The mentioned formula are obtained from the previous theorem by observing that for any $f\in C_c(G,\mathscr{C})$ one has

$$\pi\rtimes U(L_{(A,\delta_1)}f)=\int_G\pi\big([L_{(A,\delta_1)}f](x)\big)U_x\,\mathrm{d}x=\int_G\pi\big(Af(x)\big)U_x\,\mathrm{d}x=\pi(A)\pi\rtimes U(f),$$

and

$$\begin{aligned}\pi\rtimes U(L_{(\mathbf{1},\delta_x)}f)&=\int_G\pi\big([L_{(\mathbf{1},\delta_x)}f](y)\big)U_y\,\mathrm{d}y=\int_G\pi\big(\theta_xf(x^{-1}y)\big)U_y\,\mathrm{d}y\\&=\int_G U_x\pi\big(f(x^{-1}y)\big)U_x^*U_y\,\mathrm{d}y=\int_G U_x\pi\big(f(y)\big)U_y\,\mathrm{d}y=U_x\pi\rtimes U(f).\end{aligned}$$

□

Remark 3.4.7. *In the context of the previous corollary and by starting again with the double centralizer $(L_{(A,\mu)},R_{(A,\mu)})$ as above, with $A=\mathbf{1}$ and μ an element of $L^1(G)$, one can also infer that there exists a $*$-homomorphism $i_G:L^1(G)\to\mathscr{M}\big(\pi\rtimes U(\mathscr{C}\rtimes_\theta G)\big)\subset\mathscr{B}(\mathcal{H})$ such that one has $i_G(f)=\int_G f(x)i_G(x)\,\mathrm{d}x$ for any $f\in L^1(G)$. In fact, this $*$-homomorphism continuously extends to a $*$-homomorphism from $\mathscr{C}^*(G)$ to the multiplier algebra $\mathscr{M}\big(\pi\rtimes U(\mathscr{C}\rtimes_\theta G)\big)$.*

By using the multiplier algebra and the two maps introduced above, it is rather straightforward to improve Theorem 3.4.1:

Theorem 3.4.8. *For any C^*-dynamical system $(\mathscr{C},G,\theta)$, the map sending a covariant representation $(\mathcal{H},\pi,U)$ to the integrated form $(\mathcal{H},\pi\rtimes U)$ is a bijective correspondence between non-degenerate covariant representations of $(\mathscr{C},G,\theta)$ and non-degenerate representations of $\mathscr{C}\rtimes_\theta G$.*

We stress that this theorem asserts in particular that any representation of the C^*-algebra $\mathscr{C}\rtimes_\theta G$ corresponds to the integrated form of a covariant representation of the underlying dynamical system. Let us now end this section with a technical result which will be important later on. Its proof is not complicated but is based on some preliminary results which are not trivial, see Lemma 2.45 and Corollary 2.48 of [Wil07]. Note that in this section, most of the difficulties do not come from the algebraic computations but from some topological considerations.

Lemma 3.4.9. *Let $(\mathscr{C}^1, G, \theta^1)$ and $(\mathscr{C}^2, G, \theta^2)$ be C^*-dynamical systems, and let $\varphi : \mathscr{C}^1 \to \mathscr{C}^2$ be an equivariant $*$-homomorphism[3]. Then there is a $*$-homomorphism*

$$\varphi \rtimes \mathrm{id} : \mathscr{C}^1 \rtimes_{\theta^1} G \to \mathscr{C}^2 \rtimes_{\theta^2} G$$

mapping $C_c(G; \mathscr{C}^1)$ into $C_c(G; \mathscr{C}^2)$ and such that $[\varphi \rtimes \mathrm{id}(f)](x) = \varphi\big(f(x)\big)$ for any $f \in C_c(G; \mathscr{C}^1)$ and $x \in G$.

Extension 3.4.10. *Consider the special case $\mathscr{C} = C(\mathbb{T})$, $G = \mathbb{Z}$ and $[\theta_n f](z) = f(e^{i2\pi n\vartheta} z)$ for any $f \in \mathscr{C}$, $z \in \mathbb{T}$ and some fixed $\vartheta \in [0, 1]$. Depending if ϑ is rational or irrational, the corresponding algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is called* the rational or irrational rotation algebra. *Its study has been a hot topic in the early 80's, and continues to be of interest. Some preliminary information can be grasp for example in [Wil07, Prop. 2.56] and in many other references.*

3.5 Invariant ideals and crossed product

Let us consider a C^*-dynamical system $(\mathscr{C}, G, \theta)$, and let $\mathscr{J}$ be a closed and self-adjoint ideal in $\mathscr{C}$ which is θ-invariant ($\Leftrightarrow \theta_x(A) \in \mathscr{J}$ for any $A \in \mathscr{J}$ and $x \in G$). Then, each θ_x restricts to a $*$-automorphism of $\mathscr{J}$, and this defines a C^*-dynamical system $(\mathscr{J}, G, \theta)$ as well as a quotient C^*-dynamical system $(\mathscr{C}/\mathscr{J}, G, \theta)$, where

$$\theta_x(A + \mathscr{J}) = \theta_x(A) + \mathscr{J}.$$

Note that we have kept the same notation for the $*$-automorphism θ_x acting on $\mathscr{J}$ and for its action on the quotient algebra $\mathscr{C}/\mathscr{J}$. Since the inclusion map $\iota : \mathscr{J} \to \mathscr{C}$ and the quotient map $q : \mathscr{C} \to \mathscr{C}/\mathscr{J}$ are equivariant $*$-homomorphisms, they define by Lemma 3.4.9 $*$-homomorphisms $\iota \rtimes \mathrm{id} : \mathscr{J} \rtimes_\theta G \to \mathscr{C} \rtimes_\theta G$ and $q \rtimes \mathrm{id} : \mathscr{C} \rtimes_\theta G \to (\mathscr{C}/\mathscr{J}) \rtimes_\theta G$.

Clearly, $C_c(G; \mathscr{J})$ is a self-adjoint ideal in $C_c(G; \mathscr{C})$, and therefore its closure is an ideal in $\mathscr{C} \rtimes_\theta G$, which corresponds to the image of $\mathscr{J} \rtimes_\theta G$ through the $*$-homomorphism $\iota \rtimes \mathrm{id}$. In addition, it can be shown that $\iota \rtimes \mathrm{id}$ is isometric on $C_c(G; \mathscr{J})$, which implies that $\iota \rtimes \mathrm{id}$ is in fact a $*$-isomorphism onto the closure of $C_c(G; \mathscr{J})$ in $\mathscr{C} \rtimes_\theta G$, see [Wil07, Lem. 3.17] for the proof of the isometry.

Let us now state an important result about the functoriality of the crossed product:

Lemma 3.5.1. *Let $(\mathscr{C}, G, \theta)$ be a C^*-dynamical system, and let $\mathscr{J}$ be a self-adjoint closed ideal in $\mathscr{C}$ which is θ-invariant. Then we have the following short sequence of C^*-algebras:*

$$0 \longrightarrow \mathscr{J} \rtimes_\theta G \xrightarrow{\iota \rtimes \mathrm{id}} \mathscr{C} \rtimes_\theta G \xrightarrow{q \rtimes \mathrm{id}} (\mathscr{C}/\mathscr{J}) \rtimes_\theta G \longrightarrow 0.$$

[3] In the present context, the $*$-homomorphism φ is equivariant if $\varphi\big(\theta^1_x(A)\big) = \theta^2_x\big(\varphi(A)\big)$ for all $A \in \mathscr{C}^1$ and $x \in G$.

The fact that $\iota \rtimes \mathrm{id}$ is a $*$-isomorphism has already been mentioned in the paragraph preceding the statement. Thus it only remains to show that

$$\mathsf{Ker}\,\big(q \rtimes \mathrm{id}\big) = \iota \rtimes \mathrm{id}\big(\mathscr{J} \rtimes_\theta G\big)$$

which can be achieved with the use of an approximate unit, see [Wil07, Prop. 3.19] for the details.

Let us close this section by considering the previous result in the context of transformation group C^*-algebras, see Remark 3.4.4. More precisely, let us consider the C^*-dynamical system $(C_0(\Omega), G, \theta)$ with $[\theta_x f](\xi) = f(x^{-1} \cdot \xi)$ for any $f \in C_0(\Omega)$, $x \in G$ and $\xi \in \Omega$. In this framework, the θ-invariant ideals of $C_0(\Omega)$ corresponds to subalgebras $C_0(\Omega')$ with Ω' a G-invariant open subset of Ω. Then, let us set $F := \Omega \setminus \Omega'$, which is a G-invariant closed subset of Ω, and let us identify $C_0(F)$ with the quotient $C_0(\Omega)/C_0(\Omega')$ (notice that the $*$-homomorphism $q : C_0(\Omega) \to C_0(F)$ is equivariant). A special case of the previous lemma reads then:

Corollary 3.5.2. *Let us consider the C^*-dynamical system $(C_0(\Omega), G, \theta)$, and let Ω' be an open G-invariant subset of Ω. Then we have the following short exact sequence of C^*-algebras*

$$0 \longrightarrow C_0(\Omega') \rtimes_\theta G \xrightarrow{\iota\rtimes\mathrm{id}} C_0(\Omega) \rtimes_\theta G \xrightarrow{q\rtimes\mathrm{id}} C_0(\Omega \setminus \Omega') \rtimes_\theta G \longrightarrow 0. \tag{3.5.1}$$

Chapter 4

Schrödinger operators and essential spectrum

The aim of this chapter is to show how crossed product C^*-algebras can be used for the computation of some spectral information on self-adjoint operators. These operators appeared naturally in the context of quantum mechanics, but then their investigations has been developed on a pure mathematical level. For simplicity, all the following considerations will be based on the group $\mathbb{R}^d$, but with the content of the previous chapters these investigations could be made on an arbitrary locally compact abelian group. This natural generalization should hold mutatis mutandis, and it is certainly a very useful exercise to check this statement (note that the main difficulties come from the constants and from some historical conventions).

4.1 Multiplication and convolution operators

In this section, we introduce two natural classes of operators on $\mathbb{R}^d$. This material is standard and can be found for example in the books [Amr09] and [Tes09]. We start by considering multiplication operators on the Hilbert space $L^2(\mathbb{R}^d)$.

For any measurable complex function φ on $\mathbb{R}^d$ let us define the *multiplication operator* $\varphi(X)$ on $\mathcal{H} := L^2(\mathbb{R}^d)$ by

$$[\varphi(X)f](x) = \varphi(x)f(x) \qquad \forall x \in \mathbb{R}^d$$

for any

$$f \in \mathcal{D}\big(\varphi(X)\big) := \big\{g \in \mathcal{H} \mid \int_{\mathbb{R}^d} |\varphi(x)|^2 |g(x)|^2 \mathrm{d}x < \infty\big\}.$$

Clearly, the properties of this operator depend on the function φ. More precisely:

Lemma 4.1.1. *Let $\varphi(X)$ be the multiplication operator on $\mathcal{H}$. Then $\varphi(X)$ belongs to $\mathscr{B}(\mathcal{H})$ if and only if $\varphi \in L^\infty(\mathbb{R}^d)$, and in this case $\|\varphi(X)\| = \|\varphi\|_\infty$.*

Proof. One has

$$\|\varphi(X)f\|^2 = \int_{\mathbb{R}^d} |\varphi(x)|^2 |f(x)|^2 \mathrm{d}x \leq \|\varphi\|_\infty^2 \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x = \|\varphi\|_\infty^2 \|f\|^2.$$

Thus, if $\varphi \in L^\infty(\mathbb{R}^d)$, then $\mathcal{D}\big(\varphi(X)\big) = \mathcal{H}$ and $\|\varphi(X)\| \leq \|\varphi\|_\infty$.

Now, assume that $\varphi \notin L^\infty(\mathbb{R}^d)$. It means that for any $n \in \mathbb{N}$ there exists a measurable set $W_n \subset \mathbb{R}^d$ with $0 < |W_n| < \infty$ such that $|\varphi(x)| > n$ for any $x \in W_n$. We then set $f_n = \chi_{W_n}$ and observe that $f_n \in \mathcal{H}$ with $\|f_n\|^2 = |W_n|$ and that

$$\|\varphi(X)f_n\|^2 = \int_{\mathbb{R}^d} |\varphi(x)|^2 |f_n(x)|^2 \mathrm{d}x = \int_{W_n} |\varphi(x)|^2 dx > n^2 \|f_n\|^2,$$

from which one infers that $\|\varphi(X)f_n\|/\|f_n\| > n$. Since n is arbitrary, the operator $\varphi(X)$ can not be bounded.

Let us finally show that if $\varphi \in L^\infty(\mathbb{R}^d)$, then $\|\varphi(X)\| \geq \|\varphi\|_\infty$. Indeed, for any $\varepsilon > 0$, there exists a measurable set $W_\varepsilon \subset \mathbb{R}^d$ with $0 < |W_\varepsilon| < \infty$ such that $|\varphi(x)| > \|\varphi\|_\infty - \varepsilon$ for any $x \in W_\varepsilon$. Again by setting $f_\varepsilon = \chi_{W_\varepsilon}$ one gets that $\|\varphi(X)f_\varepsilon\|/\|f_\varepsilon\| > \|\varphi\|_\infty - \varepsilon$, from which one deduces the required inequality. □

If $\varphi \in L^\infty(\mathbb{R}^d)$, one easily observes that $\varphi(X)^* = \overline{\varphi}(X)$, and thus $\varphi(X)$ is self-adjoint if and only if φ is a real function. If φ is real but does not belong to $L^\infty(\mathbb{R}^d)$, one can show that the pair $\big(\varphi(X), \mathcal{D}\big(\varphi(X)\big)\big)$ defines a self-adjoint operator if $\mathcal{D}\big(\varphi(X)\big)$ is dense in $\mathcal{H}$. In particular, if $\varphi \in C(\mathbb{R}^d)$ or if $|\varphi|$ is polynomially bounded, then the mentioned operator is self-adjoint, see [Amr09, Prop. 2.29]. For example, for any $j \in \{1, \dots, d\}$ the operator X_j defined by $[X_j f](x) = x_j f(x)$ is a self-adjoint operator with domain $\mathcal{D}(X_j)$. Note that the d-tuple $(X_1, \dots, X_d)$ is often referred to as the *position operator* in $L^2(\mathbb{R}^d)$. More generally, for any $\alpha \in \mathbb{N}^d$ one also sets

$$X^\alpha = X_1^{\alpha_1} \dots X_d^{\alpha_d}$$

and this expression defines a self-adjoint operator on its natural domain. Other useful multiplication operators are defined for any $s > 0$ by the functions

$$\mathbb{R}^d \ni x \mapsto \langle x \rangle^s := \Big(1 + \sum_{j=1}^d x_j^2\Big)^{s/2} \in \mathbb{R}.$$

The corresponding operators $\big(\langle X \rangle^s, \mathcal{H}_s\big)$, with

$$\mathcal{H}_s := \Big\{ f \in \mathcal{H} \mid \langle X \rangle^s f \in \mathcal{H} \Big\} = \Big\{ f \in \mathcal{H} \mid \int_{\mathbb{R}^d} \langle x \rangle^{2s} |f(x)|^2 \mathrm{d}x < \infty \Big\},$$

are again self-adjoint operators on $\mathcal{H}$. Note that one usually calls $\mathcal{H}_s$ *the weighted Hilbert space with weight* s since it is naturally a Hilbert space with the scalar product $\langle f, g \rangle_s := \int_{\mathbb{R}^d} f(x) \overline{g(x)} \langle x \rangle^{2s} \mathrm{d}x$.

Exercise 4.1.2. *For any $\varphi \in C_b(\mathbb{R}^d)$, show that the spectrum of the multiplication operator $\varphi(X)$ coincides with the closure of $\varphi(\mathbb{R}^d)$ in $\mathbb{C}$.*

We shall now introduce a new type of operators on $\mathcal{H}$, but for that purpose we need to recall a few results about the usual Fourier transform[1] on $\mathbb{R}^d$. The Fourier transform $\mathcal{F}$ is defined on any $f \in C_c(\mathbb{R}^d)$ by the formula[2]

$$[\mathcal{F}f](\xi) \equiv \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\,\mathrm{d}x. \tag{4.1.1}$$

This linear transform maps the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ onto itself, and its inverse is provided by the formula $[\mathcal{F}^{-1}f](x) \equiv \check{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(\xi)\,\mathrm{d}\xi$. In addition, by taking Parseval's identity $\int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \mathrm{d}\xi$ into account, one obtains that the Fourier transform extends continuously to a unitary map on $L^2(\mathbb{R}^d)$. We shall keep the same notation $\mathcal{F}$ for this continuous extension, but one must be aware that (4.1.1) is valid only on a restricted set of functions.

Let us use again the multi-index notation and set for any $\alpha \in \mathbb{N}^d$

$$(-i\partial)^\alpha = (-i\partial_1)^{\alpha_1} \dots (-i\partial_d)^{\alpha_d} = (-i)^{|\alpha|} \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$$

with $|\alpha| = \alpha_1 + \dots + \alpha_d$. With this notation at hand, the following relations hold for any $f \in \mathcal{S}(\mathbb{R}^d)$ and any $\alpha \in \mathbb{N}^d$:

$$\mathcal{F}(-i\partial)^\alpha f = X^\alpha \mathcal{F} f,$$

or equivalently $(-i\partial)^\alpha f = \mathcal{F}^* X^\alpha \mathcal{F} f$. Keeping these relations in mind, one defines for any $j \in \{1, \dots, d\}$ the self-adjoint operator $D_j := \mathcal{F}^* X_j \mathcal{F}$ with domain $\mathcal{F}^*\mathcal{D}(X_j)$. Similarly, for any $s > 0$, one also defines the operator $\langle D \rangle^s := \mathcal{F}^* \langle X \rangle^s \mathcal{F}$ with domain

$$\mathcal{H}^s := \big\{ f \in \mathcal{H} \mid \langle X \rangle^s \mathcal{F} f \in \mathcal{H} \big\} \equiv \big\{ f \in \mathcal{H} \mid \langle X \rangle^s \hat{f} \in \mathcal{H} \big\}.$$

Note that the space $\mathcal{H}^s$ is called *the Sobolev space of order s*, and $(D_1, \dots, D_d)$ is usually called *the momentum operator*[3].

We can now introduce the usual *Laplace operator* $-\Delta$ acting on any $f \in \mathcal{S}(\mathbb{R}^d)$ as

$$-\Delta f = -\sum_{j=1}^d \partial_j^2 f = \sum_{j=1}^d (-i\partial_j)^2 f = \sum_{j=1}^d D_j^2 f. \tag{4.1.2}$$

---

[1] In the more general framework of arbitrary locally compact abelian group, the Fourier transform has been defined at the end of Section 3.2. The constants are chosen here such that the Fourier transform extends to a unitary map on $L^2(\mathbb{R}^d)$.

[2] Even if the group $\mathbb{R}^d$ is identified with its dual group, we will keep the notation ξ for points of its dual group.

[3] In physics textbooks, the position operator is often denoted by $(Q_1, \dots, Q_d)$ while $(P_1, \dots, P_d)$ is used for the momentum operator.

This operator admits a self-adjoint extension with domain $\mathcal{H}^2$, *i.e.* $\big(-\Delta, \mathcal{H}^2\big)$ is a self-adjoint operator in $\mathcal{H}$. However, let us stress that the expression (4.1.2) is not valid (pointwise) on all the elements of the domain $\mathcal{H}^2$. On the other hand, one has $-\Delta = \mathcal{F}^* X^2 \mathcal{F}$, with $X^2 = \sum_{j=1}^d X_j^2$, from which one easily infers that $\sigma(-\Delta) = [0, \infty)$. Indeed, this follows from the content of Exercise 4.1.2 together with the invariance of the spectrum through the conjugation by a unitary operator.

More generally, for any measurable function φ on $\mathbb{R}^d$ one sets $\varphi(D) := \mathcal{F}^*\varphi(X)\mathcal{F}$, with domain $\mathcal{D}\big(\varphi(D)\big) = \big\{f \in \mathcal{H} \mid \hat{f} \in \mathcal{D}\big(\varphi(X)\big)\big\}$, and as before this operator is self-adjoint if this domain is dense in $\mathcal{H}$, as for example for a continuous function φ or for a polynomially bounded function φ. Then, if one defines the convolution of two (suitable) functions on $\mathbb{R}^d$ by the formula

$$[k * f](x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} k(y) f(x-y)\,\mathrm{d}y$$

and if one takes the equality $\check{g} * f = \mathcal{F}^*(g\hat{f})$ into account, one infers that the operator $\varphi(D)$ corresponds to *a convolution operator*, namely

$$\varphi(D) f = \check{\varphi} * f. \tag{4.1.3}$$

Obviously, the meaning of such an equality depends on the class of functions f and g considered.

Exercise 4.1.3. *Show that the following relations hold on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$: $[iX_j, X_k] = \mathbf{0} = [D_j, D_k]$ for any $j, k \in \{1, \ldots, d\}$ while $[iD_j, X_k] = \mathbf{1}\delta_{jk}$.*

4.2 Schrödinger operators

In this section, we introduce the main operator we want to investigate.

First of all, let $h : \mathbb{R}^d \to \mathbb{R}$ be a continuous real function which diverges at infinity. Equivalently, we assume that h satisfies $(h - z)^{-1} \in C_0(\mathbb{R}^d)$ for some $z \in \mathbb{C} \setminus \mathbb{R}$. The corresponding convolution operator $h(D)$, defined by $\mathcal{F}^* h(X) \mathcal{F}$, is a self-adjoint operator with domain $\mathcal{F}^*\mathcal{D}\big(h(X)\big)$. Clearly, the spectrum of such an operator is equal to the closure of $h(\mathbb{R}^d)$ in $\mathbb{R}$.

Some examples of such a function h which are often considered in the literature are the functions defined by $h(\xi) = \xi^2$, $h(\xi) = |\xi|$ or $h(\xi) = \sqrt{1+\xi^2} - 1$. In these cases, the operator $h(D) = -\Delta$ corresponds to *the free Laplace operator*, the operator $h(D) = |D|$ is *the relativistic Schrödinger operator without mass*, while the operator $h(D) = \sqrt{1-\Delta} - 1$ corresponds to *the relativistic Schrödinger operator with mass.* In these three cases, one has $\sigma\big(h(D)\big) = \sigma_{ac}\big(h(D)\big) = [0, \infty)$ while $\sigma_{sc}\big(h(D)\big) = \sigma_p\big(h(D)\big) = \emptyset$.

Let us now perturb this operator $h(D)$ with a multiplication operator $V(X)$. If the measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is not essentially bounded, then the operator $h(D) + V(X)$ can only be defined on the intersection of the two domains, and checking that there exists a self-adjoint extension of this operator is not always an easy task. On the other hand, if one assumes that $V \in L^\infty(\mathbb{R}^d)$, then we can define the operator

$$H := h(D) + V(X) \qquad \text{with} \quad \mathcal{D}(H) = \mathcal{D}\big(h(D)\big) \tag{4.2.1}$$

and this operator is self-adjoint.

A lot of investigations have been performed on such an operator H when V vanishes at infinity, in a suitable sense. On the other hand, much less is known on this operator when the multiplication operator $V(X)$, also called *the potential*, has an anisotropic behavior. The main idea of the approach presented here is to encode the anisotropy in an algebra $\mathscr{C}$ of functions on $\mathbb{R}^d$. Then, if the potential belongs to this algebra of functions, we can show that the operator H itself belongs to the crossed product algebra. More explanations about this construction are provided in the next section.

4.3 Affiliation

The main ideas of this section are borrowed from [Măn02]. Some other references using similar ideas are [GI02, AMP02, GI06, DG13, Măn013]. From now on, we consider an algebra of functions on $\mathbb{R}^d$ satisfying the following assumptions:

Assumption 4.3.1. *$\mathscr{C}$ is a unital C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is $\mathbb{R}^d$-invariant and which contains the subalgebra $C_0(\mathbb{R}^n)$.*

Recall that this algebra is $\mathbb{R}^d$-invariant if whenever $\varphi \in \mathscr{C}$ and $x \in \mathbb{R}^d$, then $\theta_x(\varphi) := \varphi(\cdot - x) \in \mathscr{C}$. As a consequence of Theorem 2.4.15, there exists a compact space Ω such that $\mathscr{C}$ is isometrically $*$-isomorphic to $C(\Omega)$. In addition, note that from the requirement $C_0(\mathbb{R}^d) \subset \mathscr{C}$ one infers that Ω is *a compactification of* $\mathbb{R}^d$ ($\Leftrightarrow$ a compact space in which $\mathbb{R}^d$ is dense). Indeed, each point $x \in \mathbb{R}^d$ defines a distinct element of the character space Ω by setting $x \to \delta_x$ where δ_x is the evaluation at x, *i.e.* $\delta_x(\varphi) := \varphi(x)$ for any $\varphi \in \mathscr{C}$. Finally, one also observes that the action of $\mathbb{R}^d$ continuously extends to an action on Ω defined by the formula:

$$[\theta_x(\tau)](\varphi) = \tau\big(\theta_x(\varphi)\big), \tag{4.3.1}$$

for any $\varphi \in \mathscr{C}$, $x \in \mathbb{R}^d$ and $\tau \in \Omega$. Note that we use the same symbol for the action of $\mathbb{R}^d$ on itself and for its extension on Ω. In summary, the Assumptions 4.3.1 imply that the triple $\big(C(\Omega), \mathbb{R}^d, \theta\big)$ defines a C^*-dynamical system, see also Example 3.3.2 and Exercise 3.3.3.

Exercise 4.3.2. *Find a unital C^*-subalgebra of $BC_u(\mathbb{R}^n)$ which is $\mathbb{R}^d$-invariant but for which the space Ω is not a compactification of $\mathbb{R}^d$.*

Let us now construct a covariant representation of this dynamical system in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$. First of all, the algebra $C(\Omega)$ is faithfully represented in $\mathscr{B}(\mathcal{H})$ by operators of multiplication. Indeed, if one defines the $*$-homomorphism π by

$$C(\Omega) \cong \mathscr{C} \ni \varphi \mapsto \pi(\varphi) := \varphi(X) \in \mathscr{B}(\mathcal{H}),$$

then $\pi : C(\Omega) \to \mathscr{B}(\mathcal{H})$ is faithful and non-degenerate. In addition, let us consider the unitary representation of the group $\mathbb{R}^d$ on $\mathcal{H}$, namely $\{U_x\}_{x\in\mathbb{R}^d}$ given by $[U_x f](y) = f(y-x)$ for any $f \in \mathcal{H}$. With this definition, the following compatibility condition holds for any $\varphi \in \mathscr{C}$

$$\pi\big(\theta_x(\varphi)\big) = \pi\big(\varphi(\cdot - x)\big) = \varphi(X-x) = U_x\varphi(X)U_x^* = U_x\pi(\varphi)U_x^*. \tag{4.3.2}$$

As a consequence, the triple $(\mathcal{H}, \pi, U)$ defines a covariant representation of the dynamical system $\big(C(\Omega), \mathbb{R}^d, \theta\big)$, and thus a non-degenerate representation of the crossed product algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ in $\mathscr{B}(\mathcal{H})$, by Theorem 3.4.8. This representation corresponds to the integrated representation $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$.

Exercise 4.3.3. *Check that the above operator U_x is equal to the operator $e^{-ix\cdot D}$, where D is the momentum operator introduced in Section 4.1.*

In order to get a better understanding of the C^*-algebra $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$, recall that $C(\Omega)$ is unital, and therefore that $L^1(\mathbb{R}^d) \subset C(\Omega) \rtimes_\theta \mathbb{R}^d$. Thus, by applying the integrated representation $\pi \rtimes U$ defined in (3.4.1) to some $u \in L^1(\mathbb{R}^d)$, one gets

$$[\pi \rtimes U(u)f](x) = \Big[\int_{\mathbb{R}^d} u(y)U_y f \,\mathrm{d}y\Big](x) = \int_{\mathbb{R}^d} u(y)f(x-y)\,\mathrm{d}y = (2\pi)^{d/2}[\hat{u}(D)f](x),$$

where we have taken equation (4.1.3) into account. More generally, by considering products $u \otimes \varphi \subset L^1(\mathbb{R}^d) \odot C(\Omega) \subset L^1\big(\mathbb{R}^d; C(\Omega)\big)$, we get that operators of the form $(2\pi)^{d/2}\varphi(X)\hat{u}(D)$ belong to $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$. Finally, by considering linear combinations, one infers that:

Theorem 4.3.4. *Let $\mathscr{C}$ satisfy Assumption 4.3.1 and let Ω be defined by the Gelfand $*$-isomorphism $\mathscr{C} \cong C(\Omega)$. Then $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ is equal to*

$$\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle := C^*\Big(\varphi(X)v(D) \mid v \in C_0(\mathbb{R}^d) \text{ and } \varphi \in C(\Omega)\Big), \tag{4.3.3}$$

and the C^*-algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ is isometrically $*$-isomorphic to this algebra.

Proof. By construction, and since $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ is norm closed, it is quite clear that this algebra and the C^*-algebra defined in the r.h.s. of (4.3.3) are equal. However, it remains to show that the representation $\pi \rtimes U$ of $C(\Omega) \rtimes_\theta \mathbb{R}^d$ is faithful. Such a statement has been proved for example in [GI02, Thm 4.1] and is based on the regular representation introduced in Example 3.3.6. We do not provide the arguments here since we are going to prove a more general result in the context of twisted crossed product C^*-algebras in a forthcoming chapter. □

Remark 4.3.5. *In the previous statement, if one chooses[4] $C_0(\mathbb{R}^d)$ for the algebra $\mathscr{C}$, then the resulting C^*-algebra $\langle C_0(\mathbb{R}^d) \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ coincides with C^*-algebra $\mathscr{K}\big(L^2(\mathbb{R}^d)\big)$. It means that the integrated representation $\pi \rtimes U$ provides the $*$-isomorphism already mentioned in Example 3.4.3.*

[4]Obviously, $C_0(\mathbb{R}^d)$ is not unital, but this lack of a unity can easily be taken into account in the previous construction.

We are now in a suitable position for explaining the link between the Schrödinger operator H and the C^*-algebra introduced in (4.3.3).

Lemma 4.3.6. *Let $\mathscr{C}$ satisfy Assumption 4.3.1 and let Ω be defined by the Gelfand $*$-isomorphism $\mathscr{C} \cong C(\Omega)$. Let $h \in C(\mathbb{R}^d;\mathbb{R})$ be diverging at infinity, let $V \in C(\Omega;\mathbb{R})$, and let $H := h(D) + V(X)$. Then, for some $z \in \mathbb{C} \setminus \mathbb{R}$ with $|\Im z|$ large enough, the resolvent $(H-z)^{-1}$ belongs to the C^*-algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$.*

Proof. Let us consider the Neumann series

$$\begin{aligned}
(H-z)^{-1} &= \big(h(D) - z + V(X)\big)^{-1} \\
&= \big(h(D)-z\big)^{-1}\Big(\mathbf{1} + V(X)\big(h(D)-z\big)^{-1}\Big)^{-1} \\
&= \big(h(D)-z\big)^{-1}\sum_{n=0}^{\infty}(-1)^n\big[V(X)\big(h(D)-z\big)^{-1}\big]^n,
\end{aligned}$$

where we have used the result of Lemma 4.1.1 and suitably chosen z such that

$$\big\|V(X)\big(h(D)-z\big)^{-1}\big\| < 1.$$

Since each term in the series belongs to $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, and since the series converges in norm of $\mathscr{B}(\mathcal{H})$, it follows that the series converges in $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. □

Note that from the resolvent equation (1.6.1), one infers the equalities

$$\begin{aligned}
(H-z)^{-1} &= (H-z_0)^{-1}\big(\mathbf{1} + (z-z_0)(H-z_0)^{-1}\big)^{-1} \\
&= \sum_{n=0}^{\infty}(z-z_0)^n\big((H-z_0)^{-1}\big)^{n+1}.
\end{aligned}$$

By starting then from the result of the previous lemma and by a approximation argument, one deduces that if $(H-z_0)^{-1} \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$, then $(H-z)^{-1} \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. By a density argument, it even follows that $\varphi(H) \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for any $\varphi \in C_0(\mathbb{R})$, and the operator $\varphi(H)$ corresponds to the one also mentioned in Definition 1.7.9. Thus, H defines a $*$-homomorphism from $C_0(\mathbb{R})$ to $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. More generally, one sets:

Definition 4.3.7. *(i)* An observable affiliated to a C^*-algebra $\mathfrak{C}$ *is a $*$-homomorphism $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.*

(ii) The spectrum $\sigma(\Phi)$ of an observable Φ *consists in the set of $\lambda \in \mathbb{R}$ such that $\Phi(\varphi) \neq \mathbf{0}$ whenever $\varphi \in C_0(\mathbb{R})$ and $\varphi(\lambda) \neq 0$.*

Let us stress that for the previous definition of an observable, the C^*-algebra $\mathfrak{C}$ does not need to be represented in a Hilbert space. On the other hand, with the observation made just before the definition, one observes that if H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, and if $\mathfrak{C}$ is a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ with $(H-z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C} \setminus \mathbb{R}$, then H defines an observable affiliated to $\mathfrak{C}$, which we denote by Φ^H (in this case one has $\Phi^H(\varphi) = \varphi(H)$).

Exercise 4.3.8. *In the framework of the previous paragraph, show that $\sigma\big(\Phi^H\big) = \sigma(H)$.*

4.4 $\mathfrak{J}$-essential spectrum

Let us consider a C^*-algebra $\mathfrak{C}$ and one ideal $\mathfrak{J}$ in $\mathfrak{C}$ (in this section ideals of C^*-algebras will always be considered closed and self-adjoint). By Corollary 2.5.6, the quotient algebra $\mathfrak{C}/\mathfrak{J}$ is a C^*-algebra, and let us denote by $q : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}$ the quotient $*$-homomorphism. Then, if Φ is an observable affiliated to $\mathfrak{C}$, the composed map $q \circ \Phi : C_0(\mathbb{R}) \to \mathfrak{C}/\mathfrak{J}$ defines an observable affiliated to the quotient algebra. In this setting, one has:

Definition 4.4.1. *Let $\mathfrak{C}$ be a C^*-algebra, with $\mathfrak{J}$ an ideal in $\mathfrak{C}$, and let Φ be an observable affiliated to $\mathfrak{C}$. The spectrum $\sigma(q \circ \Phi)$ of the observable $q \circ \Phi$ is called* the $\mathfrak{J}$-essential spectrum *of Φ and will be denoted by $\sigma^{\mathfrak{J}}(\Phi)$,* i.e. $\sigma^{\mathfrak{J}}(\Phi) = \sigma(q \circ \Phi)$.

Exercise 4.4.2. *In the framework of the previous definition, show that $\lambda \in \sigma^{\mathfrak{J}}(\Phi)$ if and only if $\Phi(\varphi) \notin \mathfrak{J}$ whenever $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) \neq 0$.*

To motivate the introduction of this notion of $\mathfrak{J}$-essential spectrum, let us derive the original result in this setting:

Lemma 4.4.3. *Let $\mathcal{H}$ be a Hilbert space, and H be a self-adjoint operator on $\mathcal{H}$. Then the following equality holds:*

$$\sigma_{ess}(H) = \sigma^{\mathscr{K}(\mathcal{H})}\big(\Phi^H\big),$$

or in other words, the essential spectrum of H can be computed by considering the $\mathscr{K}(\mathcal{H})$-essential spectrum of the corresponding observable affiliated to $\mathscr{B}(\mathcal{H})$.

Proof. From the definition of the essential spectrum provided in Definition 1.7.17, it is easily observed that $\lambda \in \sigma_{ess}(H)$ if and only if $E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H}$ is infinite dimensional for any $\delta > 0$, where $E(\cdot)$ denotes the spectral measure associated with the self-adjoint operator H, see Section 1.7.2. This property in then equivalent to the fact that if $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) > 0$, the corresponding operator $\Phi^H(\varphi) = \varphi(H) \notin \mathscr{K}(\mathcal{H})$. Indeed:

$\Leftarrow$: let $\delta > 0$ and choose $\varphi \in C_c\big((\lambda-\delta, \lambda+\delta)\big)$ with $\varphi(\lambda) > 0$. By assumption $\varphi(H) \notin \mathscr{K}(\mathcal{H})$, and therefore $E\big((\lambda-\delta, \lambda+\delta)\big) \notin \mathscr{K}(\mathcal{H})$ since otherwise one would have $\varphi(H) = \varphi(H)E\big((\lambda-\delta, \lambda+\delta)\big) \in \mathscr{K}(\mathcal{H})$.

$\Rightarrow$: By absurd let us assume that there exists $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) > 0$ such that $\varphi(H) \in \mathscr{K}(\mathcal{H})$. Therefore, for any $\varepsilon > 0$ with $\varphi(\lambda)/2 > \varepsilon$ there exist $\{g_j, h_j\}_{j=1}^n \subset \mathcal{H}$ such that $\|\varphi(H) - A_n\| < \varepsilon$, see equation (1.3.1) for the definition of A_n. Then, let us choose $\delta > 0$ such that $\varphi(\lambda') > \varphi(\lambda) - \varepsilon$ for any $\lambda' \in (\lambda-\delta, \lambda+\delta)$. By assumption, the subspace $E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H}$ is infinite dimensional, and so is the subspace

$$E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H} \cap \mathsf{Vect}\,\big(\{g_j, h_j \mid j \in \{1, \ldots, n\}\}\big)^{\perp}.$$

It finally follows from Proposition 1.7.4.(iii) for any f in the above set one has

$$\|\varphi(H)f\|^2 = \int_{\lambda-\delta}^{\lambda+\delta} |\varphi(\lambda')|^2 m_f(\mathrm{d}\lambda') > (\varphi(\lambda)-\varepsilon)^2 \int_{\lambda-\delta}^{\lambda+\delta} m_f(\mathrm{d}\lambda') = (\varphi(\lambda)-\varepsilon)^2\|f\|^2,$$

implying that $\|\varphi(H)f\| > (\varphi(\lambda)-\varepsilon)\|f\| > \varphi(\lambda)\|f\|/2 > \varepsilon\|f\|$. However, this estimate contradicts the initial assumption which stated that

$$\|\varphi(H)f\| = \big\|\big(\varphi(H)-A_n\big)f\big\| < \varepsilon\|f\|.$$

□

Now, if $\mathfrak{J}$ is an ideal in a C^*-algebra $\mathfrak{C}$, the computation of the $\mathfrak{J}$-essential spectrum of an observable Φ affiliated to $\mathfrak{C}$ can sometimes be eased by the existence of a larger family of ideals $\mathfrak{J}_i$ in $\mathfrak{C}$ which satisfy $\cap_i\mathfrak{J}_i = \mathfrak{J}$. Our interest in such a family is that the quotient algebras $\mathfrak{C}/\mathfrak{J}_i$ might be more easily understandable than the quotient $\mathfrak{C}/\mathfrak{J}$. Note that in this framework we shall denote by q the quotient map $\mathfrak{C}\to\mathfrak{C}/\mathfrak{J}$ and by q_i the quotient map $\mathfrak{C}\to\mathfrak{C}/\mathfrak{J}_i$. Our next aim is to show that with such a construction, the spectral properties are preserved. Note that we shall use the notation $\hookrightarrow$ for injective $*$-homomorphisms.

Proposition 4.4.4. *Let $\mathfrak{C}$ be a C^*-algebra, and $\mathfrak{J},\mathfrak{J}_i$ be ideals in $\mathfrak{C}$ satisfying $\cap_i\mathfrak{J}_i=\mathfrak{J}$.*

(i) There is a canonical injective $*$-homomorphism $\mathfrak{C}/\mathfrak{J}\hookrightarrow \Pi_i\ \mathfrak{C}/\mathfrak{J}_i$,

(ii) If Φ is an observable affiliated to $\mathfrak{C}$, and if one sets $\Phi_i := q_i\circ\Phi$ for the observable affiliated to $\mathfrak{C}/\mathfrak{J}_i$, then one has

$$\sigma^{\mathfrak{J}}(\Phi) = \overline{\cup_i\ \sigma(\Phi_i)} \tag{4.4.1}$$

Proof. With the notation introduced before the statement, one has that the kernel of q_i is $\mathfrak{J}_i$. Thus, the kernel of $(q_i)_i:\mathfrak{C}\to\Pi_i\ \mathfrak{C}/\mathfrak{J}_i$ is $\cap_i\mathfrak{J}_i=\mathfrak{J}$.

By definition, one has

$$\begin{aligned}\sigma^{\mathfrak{J}}(\Phi) &= \sigma(q\circ\Phi)\\ &= \big\{\lambda\in\mathbb{R}\mid q\big(\Phi(\varphi)\big)\neq\mathbf{0}\ \forall\varphi\in C_0(\mathbb{R})\text{ with }\varphi(\lambda)\neq 0\big\}\\ &= \overline{\cup_i\big\{\lambda\in\mathbb{R}\mid q_i\big(\Phi(\varphi)\big)\neq\mathbf{0}\ \forall\varphi\in C_0(\mathbb{R})\text{ with }\varphi(\lambda)\neq 0\big\}}\\ &= \overline{\cup_i\sigma(\Phi_i)}.\end{aligned}$$

Alternatively, we can use that for any $\varphi\in C_0(\mathbb{R})$ one has

$$\sigma\big(q\circ\Phi(\varphi)\big) = \sigma\big[q\big(\Phi(\varphi)\big)\big] = \sigma\big[\Pi_i q_i\big(\Phi(\varphi)\big)\big] = \overline{\cup_i\sigma\big(\Phi_i(\varphi)\big)}$$

where we have used that the spectrum is invariant under an injective $*$-homomorphism[5] and that the spectrum of an operator belonging to a direct product is the closure of the union of the spectrum of its components. □

[5]Indeed, if $\mathscr{C}$, $\mathscr{Q}$ are C^*-algebras and if $\varphi:\mathscr{C}\to\mathscr{Q}$ is an injective $*$-homomorphism, it follows from Corollary 2.5.8 that $\mathscr{C}$ and $\varphi(\mathscr{C})\subset\mathscr{Q}$ are isometrically $*$-isomorphic, and thus computing the spectrum of $A\in\mathscr{C}$ or of $\varphi(A)\in\mathscr{Q}$ provides the same result.

Remark 4.4.5. *In the above framework, if $\mathfrak{C} = \mathscr{B}(\mathcal{H})$ and if $\mathfrak{J} = \mathscr{K}(\mathcal{H})$ then the quotient algebra $\mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ is called* the Calkin algebra. *In this situation, there does not exist any other ideal in $\mathscr{B}(\mathcal{H})$, and thus the construction provided in the previous proposition is useless. However, if $\mathfrak{C}$ is a C^*-subalgebra smaller than $\mathscr{B}(\mathcal{H})$ but with $\mathscr{K}(\mathcal{H}) \subset \mathfrak{C}$, then the above construction might provide lots of information, as we shall see in the following section.*

4.5 Orbits and essential spectrum

Our aim in this section is to compute the essential spectrum of the operator $H = h(D)+V(X)$, with $h : \mathbb{R}^d \to \mathbb{R}$ a continuous real function which diverges at infinity, and with $V \in \mathscr{C}$, this C^*-algebra satisfying itself Assumptions 4.3.1. Since by Lemma 4.3.6 the operator H corresponds to an observable affiliated to the C^*-algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ defined in (4.3.3), and since by Remark 4.3.5 we already know that $\mathscr{K}(\mathcal{H}) \subset \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, Proposition 4.4.4 encourages us to find a suitable family of other ideals of $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ surrounding $\mathscr{K}(\mathcal{H})$ in the sense of that proposition. Thanks to the functoriality of the crossed product, as presented in Corollary 3.5.2, these investigations can be performed quite easily at an abelian level.

Recall first that Ω is a compactification of $\mathbb{R}^d$. In addition, the group $\mathbb{R}^d$ acts continuously on Ω, and we use the notation $\theta_x(\tau)$ for the action of $x \in \mathbb{R}^d$ on $\tau \in \Omega$, see also Exercise 3.3.3. Clearly, $\mathbb{R}^d$ is an open and $\mathbb{R}^d$-invariant subset of Ω, and therefore $C_0(\mathbb{R}^d)$ corresponds to a $\mathbb{R}^d$-invariant ideal of $C(\Omega)$, see the end of Section 3.5. Now, if we denote by $\partial\Omega$ the boundary of Ω ($\Leftrightarrow \Omega \setminus \mathbb{R}^d$), then $\partial\Omega$ is a closed $\mathbb{R}^d$-invariant subset of Ω. By taking into account corollary 3.5.2 we deduce the existence of the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}^d) \rtimes_\theta \mathbb{R}^d \longrightarrow C(\Omega) \rtimes_\theta \mathbb{R}^d \longrightarrow C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \longrightarrow 0. \tag{4.5.1}$$

Note that by Theorem 4.3.4 and Remark 4.3.5 we already know faithful representations of the first two algebras in the Hilbert space $\mathcal{H}$. Our aim is thus to obtain a better understanding of the third algebra, by decomposing it into suitable components.

Definition 4.5.1. *Let (Ω, G, θ) be a locally compact transformation group, and let $\tau \in \Omega$.* The orbit *$\mathcal{O}_\tau$ of τ is the set $\{\theta_x(\tau) \mid x \in G\}$, while the quasi-orbit $\mathcal{Q}_\tau$ of τ corresponds to the closure of $\mathcal{O}_\tau$ in Ω.*

Clearly, each orbit and each quasi-orbits are $\mathbb{R}^d$-invariant subsets of Ω. In addition, observe that if $\tau \in \mathbb{R}^d \subset \Omega$, then $\mathcal{O}_\tau$ is a dense orbit in Ω, and therefore $\mathcal{Q}_\tau = \Omega$. On the other hand, if we choose $\tau \in \Omega \setminus \mathbb{R}^d$, then $\mathcal{O}_\tau \subset \partial\Omega$ and $\mathcal{Q}_\tau$ is therefore a closed subset of $\partial\Omega$. Remark however that the set of all quasi-orbits is not a partition of Ω, since quasi-orbits may overlap or there may even be a strict inclusion between them. For that reason, a quasi-orbit is said *maximal* if it is not strictly contained in some other quasi-orbit. On the other hand, note that a subset Ω' of Ω is *minimal* if this set is non-empty, closed and invariant and if no proper subset of Ω' has these three properties. For example, a quasi-orbit is minimal if it does not contain any other proper quasi-orbit. Note that any quasi-orbit contains a minimal one (it may be the quasi-orbit itself).

Exercise 4.5.2. *For any $\tau \in \partial\Omega$ and for any $f \in C(\mathcal{Q}_\tau)$, check that the map*

$$\mathbb{R}^d \ni x \mapsto f\big(\theta_x(\tau)\big) \in \mathbb{C} \tag{4.5.2}$$

is an element of $BC_u(\mathbb{R}^d)$, and that the map $f \mapsto f\big(\theta_\cdot(\tau)\big)$ is injective. Note that from now on and with a slight abuse of notation, we shall always identify $C(\mathcal{Q}_\tau)$ with its realization as a subalgebra of $BC_u(\mathbb{R}^d)$, as prescribed by (4.5.2).

Let us now consider $\{\mathcal{Q}_{\tau_i}\}_i$ a covering of $\partial\Omega$ by quasi-orbits. Clearly, it implies the existence of an injective $*$-homomorphism

$$\varphi : C(\partial\Omega) \ni f \hookrightarrow (f_i)_i \in \Pi_i\ C(\mathcal{Q}_{\tau_i}),$$

where f_i corresponds to the restriction of f to $\mathcal{Q}_{\tau_i}$. Note that this morphism is rarely surjective, but that the following condition holds, namely

$$\lim_{x\to 0} \sup_i \|\theta_x^i(f_i) - f_i\| = \lim_{x\to 0} \sup_i \big\|\theta_x^i(f|_{\mathcal{Q}_{\tau_i}}) - f|_{\mathcal{Q}_{\tau_i}}\big\| = \lim_{x\to 0} \|\theta_x(f) - f\| = 0. \tag{4.5.3}$$

Here θ_x^i denotes the restriction of θ_x to $\mathcal{Q}_{\tau_i}$. In order to keep track of the property (4.5.3), we denote by $\Pi_i' C(\mathcal{Q}_{\tau_i})$ the C^*-subalgebra of $\Pi_i C(\mathcal{Q}_{\tau_i})$ on which this continuity property holds. From these considerations, one infers that $\big(\Pi_i' C(\mathcal{Q}_{\tau_i}), \mathbb{R}^d, \Pi_i\theta^i\big)$ is a C^*-dynamical system and that

$$\varphi' : C(\partial\Omega) \ni f \hookrightarrow (f_i)_i \in \Pi_i'\ C(\mathcal{Q}_{\tau_i})$$

is an equivariant $*$-homomorphism. Then, by the functoriality of the crossed product (see Lemma 3.4.9), one infers that

$$C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \hookrightarrow \Big(\Pi_i'\ C(\mathcal{Q}_{\tau_i})\Big) \rtimes_{\Pi_i\theta^i} \mathbb{R}^d \hookrightarrow \Pi_i\big(C(\mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d\big),$$

where we have taken into account the identification of $C(\mathcal{Q}_{\tau_i})$ with a C^*-subalgebra of $BC_u(\mathbb{R}^d)$ as mentioned in Exercise 4.5.2.

By summarizing our findings, one has obtained that

$$\begin{aligned}
\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \big/ \mathscr{K}\big(L^2(\mathbb{R}^d)\big) &= \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \big/ \langle C_0(\mathbb{R}^d) \cdot C_0(\hat{\mathbb{R}}^d)\rangle \\
&\cong \mathscr{C} \rtimes_\theta \mathbb{R}^d \big/ C_0(\mathbb{R}^d) \rtimes_\theta \mathbb{R}^d \\
&\cong C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \\
&\hookrightarrow \Big(\Pi_i'\ C(\mathcal{Q}_{\tau_i})\Big) \rtimes_{\Pi_i\theta^i} \mathbb{R}^d \\
&\hookrightarrow \Pi_i\ \big(C(\mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d\big) \\
&\cong \Pi_i\ \langle C(\mathcal{Q}_{\tau_i}) \cdot C_0(\hat{\mathbb{R}}^d)\rangle.
\end{aligned} \tag{4.5.4}$$

We shall denote by ι_{ess} the resulting injective $*$-homomorphism.

Remark 4.5.3. *Note that the same result would have been obtained if we had considered the ideals $C_0(\Omega \setminus \mathcal{Q}_{\tau_i})$ of $C(\Omega)$, and observed that $\cap_i\, C_0(\Omega \setminus \mathcal{Q}_{\tau_i}) = C_0(\mathbb{R}^d)$. Then, by identifying in Proposition 4.4.4 the algebra $\mathfrak{C}$ with $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and the ideals $\mathfrak{J}_i$ with $C_0(\Omega \setminus \mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d$, the first statement of this proposition would have coincide with the above result.*

We can now state the main result of this section:

Theorem 4.5.4. *Let $H = h(D) + V(X)$ be the self-adjoint operator defined in Lemma 4.3.6. Let $\{\mathcal{Q}_{\tau_i}\}_i$ be a covering of $\partial\Omega$ by quasi-orbits, and let us set $V_i := V\big(\theta.(\tau_i)\big) \in BC_u(\mathbb{R}^d)$ and $H_i := h(D) + V_i(X)$. Then*

$$\sigma_{ess}(H) = \overline{\cup_i\, \sigma(H_i)}. \tag{4.5.5}$$

Proof. In fact, most of the proof has already been performed before the statement. Indeed, by Lemma 4.3.6 we already know that H defines an observable Φ^H affiliated to the algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. Then, by keeping track of the form of all the $*$-homomorphisms, we see that ι_{ess} transforms the class modulo $\mathscr{K}(\mathcal{H})$ of the element $V(X)\big(h(D) - z\big)^{-1}$ into $\Big(V_i(X)\big(h(D) - z\big)^{-1}\Big)_i$. Thus, if q denotes the map

$$q : \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \to \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \Big/ \mathscr{K}\big(L^2(\mathbb{R}^d)\big)$$

then by taking the Neumann series into account, one deduces that $\iota_{ess}(q \circ \Phi^H) = \big(\Phi^{H_i}\big)_i$. Finally, since the spectrum is invariant under an injective $*$-homomorphism and since the spectrum of an operator belonging to a direct product is the closure of the union of the spectrum of its components, one directly gets

$$\sigma_{ess}(H) = \sigma(q \circ \Phi^H) = \overline{\cup_i\, \sigma(\Phi^{H_i})} = \overline{\cup_i\, \sigma(H_i)}.$$

□

Note that this result should be compared with the content of Proposition 1.7.18. Note also that such a result holds for more general observables affiliated to the algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, but stronger affiliation criteria are then necessary.

In the publications [Măn02] and [AMP02], highly non-trivial applications of the previous result have been presented. In part of these examples, the index i belongs to a continuum, and the corresponding result could hardly be guessed by constructing Weyl sequences, as introduced in Proposition 1.7.18. On the other hand, let us present a situation which is much more tractable, see [Ric05] for details.

Example 4.5.5 (Cartesian anisotropy). *Let $\mathscr{C}$ be the C^*-algebra made of functions on $\mathbb{R}^2$ having* a cartesian anisotropy, i.e. *$V \in \mathscr{C}$ if and only if there exists $V_1^\pm$, $V_2^\pm$ in $BC_u(\mathbb{R})$ such that*

$$\lim_{x\to\pm\infty} \sup_{y\in\mathbb{R}} \big|V(x,y) - V_2^\pm(y)\big| = 0 \quad \textit{and} \quad \lim_{y\to\pm\infty} \sup_{x\in\mathbb{R}} \big|V(x,y) - V_1^\pm(x)\big| = 0.$$

In this case, the compact space Ω *is rather easy to describe, namely*

$$\Omega = [-\infty, \infty] \times [-\infty, \infty],$$

and if one sets $H_j^{\pm} = h(D) + V_j^{\pm}(X)$, *then* (4.5.5) *reads:*

$$\sigma_{ess}(H) = \sigma(H_1^+) \cup \sigma(H_1^-) \cup \sigma(H_2^+) \cup \sigma(H_2^-).$$

Exercise 4.5.6. *Consider the cartesian anisotropy in an arbitrary dimension, as introduced in Section 3 of [Ric05].*

In the previous example, the space Ω was easily understandable. However, even if Ω is not so explicit, computations can be performed based on our understanding of quasi-orbits. We present a final example in this direction.

Example 4.5.7 (Vanishing oscillations)**.** *Let us consider the* C^**-algebra* $\mathscr{C} = VO(\mathbb{R}^d)$ *of functions with vanishing oscillations,* i.e. $V \in VO(\mathbb{R}^d)$ *if and only if* $V \in C_b(\mathbb{R}^d)$ *and for any* $x \in \mathbb{R}^d$, *the function* $V(\cdot - x) - V(\cdot)$ *belongs to* $C_0(\mathbb{R}^d)$. *Clearly,* $VO(\mathbb{R}^d)$ *is a unital* $\mathbb{R}^d$*-invariant* C^**-subalgebra of* $BC_u(\mathbb{R}^d)$, *and contains* $C_0(\mathbb{R}^d)$. *Therefore,* Ω *is a compactification of* $\mathbb{R}^d$, *and each point of* $\partial\Omega$ *is an orbit in itself. Indeed, if* $\tau \in \partial\Omega$, *then* $\tau(\varphi) = 0$ *for any* $\varphi \in C_0(\mathbb{R}^d)$, *and then by* (4.3.1) *one has for any* $x \in \mathbb{R}^d$ *and* $\varphi \in VO(\mathbb{R}^d)$*:*

$$[\theta_x(\tau)](\varphi) = \tau\big(\varphi(\cdot - x)\big) = \tau\big(\varphi(\cdot - x) - \varphi(\cdot)\big) + \tau\big(\varphi(\cdot)\big) = \tau(\varphi).$$

Thus, the only covering of $\partial\Omega$ *is obtained by* $\{\tau_i\}_{i\in\partial\Omega}$, *the asymptotic potentials are just constants, and in this case* (4.5.5) *reads:*

$$\sigma_{ess}(H) = \overline{\cup_{i\in\partial\Omega}\, \sigma\big(h(D) + V|_{\tau_i}\big)} = \big[\min h + \min V(\mathbb{R}^d)_{asy}, \infty\big)$$

where $V(\mathbb{R}^d)_{asy} := \cap_K \overline{V(\mathbb{R}^d \setminus K)}$ *and* K *are arbitrary compact neighbourhoods of* 0 *in* $\mathbb{R}^d$.

Chapter 5

Twisted crossed product C^*-algebras

This chapter is mainly dedicated to a brief introduction on twisted C^*-dynamical systems, twisted crossed products and on their representations. We mainly follow the survey article [MPR05] which is based on the standard references [BS70, Pac94, PR89, PR90]. To simplify, we undertake various hypotheses which are not needed for part of the arguments. Primarily, we assume that an *abelian* locally compact group acts upon an *abelian* C^*-algebra. This will allow us to use the Fourier transform and the Gelfand theory. Note that the general framework can easily be guessed from Section 3.1 on locally compact groups and from Section 3.4 on crossed product C^*-algebras. Note also that from now on, the additive notation will be used for the group, since in the applications we shall mainly consider the group $\mathbb{R}^d$.

To make the transition towards pseudodifferential operators and the magnetic case, we introduce at the end of the chapter a special type of twisted crossed products, in which the algebra is composed of continuous functions defined on the group. It is preceded and prepared by some considerations in group cohomology.

5.1 Twisted C^*-dynamical systems

Let us directly start with the definition of twisted dynamical systems. This definition corresponds to a generalization of Definition 3.3.1 in which no twist was introduced.

Definition 5.1.1. An (abelian) twisted C^*-dynamical system *consists in a quadruplet* $(\mathscr{C}, G, \theta, \omega)$*, where* $\mathscr{C}$ *is an abelian* C^**-algebra,* G *is a locally compact abelian group,* $\theta : G \to \mathsf{Aut}(\mathscr{C})$ *is a continuous homomorphism from* G *to the group of* $*$*-automorphisms of* $\mathscr{C}$ *(endowed with the pointwise convergence topology), and* ω *is a strictly continuous normalized* 2*-cocycle on* G *with values in the unitary group of the multiplier algebra of* $\mathscr{C}$*.*

Note that the pair (θ, ω) is often called *a twisted action of* G *on* $\mathscr{C}$. Very often, we shall use the shorter expression *twisted dynamical system* for the quadruplet $(\mathscr{C}, G, \theta, \omega)$.

Remark 5.1.2. *(i) Almost everything in this section would be true, with only some minor modifications, without assuming $\mathscr{C}$ and G to be abelian. However, our main interest lies in the connection between twisted dynamical systems and pseudodifferential theories. And for this purpose commutativity is extremely useful, almost essential. Therefore we do assume it from the very beginning.*

(ii) A strictly continuous 2-cocycle is a function $\omega : G \times G \to \mathscr{U}(\mathscr{C})$ (the unitary group in the multiplier algebra $\mathscr{M}(\mathscr{C})$ of $\mathscr{C}$), continuous with respect to the strict topology on $\mathscr{U}(\mathscr{C})$, and such that for all $x, y, z \in G$:

$$\omega(x+y,z)\,\omega(x,y) = \theta_x[\omega(y,z)]\,\omega(x,y+z). \tag{5.1.1}$$

We shall also assume it to be normalized:

$$\omega(x,0) = \omega(0,x) = 1, \qquad \text{for all } x \in G. \tag{5.1.2}$$

It is known that any automorphism of $\mathscr{C}$ extends uniquely to a $*$-automorphism of $\mathscr{M}(\mathscr{C})$ and, obviously, leaves $\mathscr{U}(\mathscr{C})$ invariant. By applying this fact to θ_x and by denoting the extension with the same symbol, one gives a sense to (5.1.1). Actually, by suitable particularizations in (5.1.1), we get $\theta_{-x}[\omega(x,0)] = \omega(0,0) = \omega(0,x)$, $\forall x \in G$, hence for normalization it suffices to ask $\omega(0,0) = 1$. The required continuity (see Definition 2.5.14) can be rephrased in this abelian setting by saying that for any $\varphi \in \mathscr{C}$, the map

$$G \times G \ni (x,y) \mapsto \varphi\omega(x,y) \in \mathscr{C}$$

is continuous. In fact Borel conditions could be imposed instead of continuity for most of the constructions and results; we do not pursue this here.

(iii) Since $\mathscr{C}$ is abelian, we know by Gelfand theory that there exists a locally compact space Ω such that $\mathscr{C}$ is isometrically $*$-isomorphic to $C_0(\Omega)$, i.e. $\mathscr{C} \cong C_0(\Omega)$. If the C^*-algebra $C_0(\Omega)$ is not unital, then $C_b(\Omega)$, the C^*-algebra of all bounded and continuous complex functions on Ω, surely is. It contains $C_0(\Omega)$ as an essential ideal. In fact $C_b(\Omega)$ can be identified with the multiplier algebra $\mathscr{M}(\mathscr{C})$ of $\mathscr{C}$. Thus the unitary group of $\mathscr{C}$ is identified with $C(\Omega;\mathbb{T})$, the family of all continuous functions on Ω taking values in the group $\mathbb{T}$ of complex numbers of modulus 1. Moreover, the strict topology on $C(\Omega;\mathbb{T})$ coincides with the topology of uniform convergence on compact subsets of Ω.

We can now go on with covariant representations, by slightly adapting Definition 3.3.4.

Definition 5.1.3. A covariant representation *of an (abelian) twisted C^*-dynamical system $(\mathscr{C}, G, \theta, \omega)$ consists in a triple $(\mathcal{H}, \pi, U)$, where*

(i) $(\mathcal{H}, \pi)$ is a (non-degenerate) representation of $\mathscr{C}$,

(ii) $(\mathcal{H}, U)$ *is a strongly continuous map from* G *to* $\mathscr{U}(\mathcal{H})$ *which satisfies*

$$U_x U_y = \pi\big(w(x,y)\big) U_{x+y} \qquad \forall x, y \in G, \tag{5.1.3}$$

(iii) the following compatibility condition holds

$$\pi\big(\theta_x(\varphi)\big) = U_x \pi(\varphi) U_x^* \qquad x \in G, \varphi \in \mathscr{C}. \tag{5.1.4}$$

One observes that in this framework U is a sort of generalized projective representation of G. The usual notion of projective representation corresponds to the case in which for all $x, y \in G$, $\omega(x,y) \in \mathbb{T}$, *i.e.* $\omega(x,y)$ is a constant function on the spectrum Ω of $\mathscr{C}$.

For twisted C^*-dynamical systems, regular representations also exist, see Example 3.3.6 in the context of dynamical systems without twist. We present below the construction borrowed from Definition 3.10 of [PR89] (note that the conventions are slightly different from Example 3.3.6 since here the right action is used instead of the left action, but these modifications are not really relevant).

Example 5.1.4 (Regular representation)**.** *Let* $(\mathscr{C}, G, \theta, \omega)$ *be an (abelian) twisted* C^**-dynamical system, and let* $(\mathcal{H}, \pi)$ *be a faithful representation of* $\mathscr{C}$*. Consider the Hilbert space* $\tilde{\mathcal{H}} := L^2(G;\mathcal{H})$*, and define* $\tilde\pi : \mathscr{C} \to \mathscr{B}(\tilde{\mathcal{H}})$ *and* $\tilde U : G \to \mathscr{U}(\tilde{\mathcal{H}})$ *by*

$$[\tilde\pi(\varphi)h](x) := \pi\big(\theta_x(\varphi)\big)h(x) \qquad \text{and} \qquad [\tilde U_y h](x) := \pi\big(\omega(x,y)\big)h(x+y), \tag{5.1.5}$$

for any $\varphi \in \mathscr{C}$*,* $h \in \tilde{\mathcal{H}}$ *and* $x, y \in G$*. It is then checked straightforwardly that the triple* $\big(\tilde{\mathcal{H}}, \tilde\pi, \tilde U\big)$ *is a covariant representation of the (abelian) twisted* C^**-dynamical system.*

Exercise 5.1.5. *Check carefully the statements contained in the previous example.*

5.2 Twisted crossed product algebras

Let $(\mathscr{C}, G, \theta, \omega)$ be an (abelian) twisted dynamical system. As for the non-twisted case, we start by mixing together the algebra $\mathscr{C}$ and the space $C_c(G)$ in a way to form a *-algebra. We define $C_c(G;\mathscr{C})$, the set of compactly supported $\mathscr{C}$-valued functions, and endow it with the norm $\|f\|_1 := \int_G \|f(x)\| \mathrm{d}x$. Let us also fix an element τ of the set $\mathsf{End}(G)$ of continuous endomorphisms of G. Particular cases are $\mathbf{0}, \mathbf{1} \in \mathsf{End}(G)$, $\mathbf{0}(x) := 0$ and $\mathbf{1}(x) := x$, for all $x \in G$. Addition and subtraction of endomorphisms are well-defined. For elements f, g of $C_c(G;\mathscr{C})$ and for any point $x \in G$ we set

$$(f *^\omega_\tau g)(x) := \int_G \theta_{\tau(y-x)}\,[f(y)]\;\theta_{(\mathbf{1}-\tau)y}\,[g(x-y)]\;\theta_{-\tau x}\,[\omega(y, x-y)]\,\mathrm{d}y \tag{5.2.1}$$

and

$$f^{*^\omega_\tau}(x) := \theta_{-\tau x}[\omega(x,-x)^{-1}]\,\theta_{(\mathbf{1}-2\tau)x}\left[\overline{f(-x)}\right], \tag{5.2.2}$$

where $\overline{f(-x)}$ corresponds to the involution of $\mathscr{C}$ applied to $f(-x)$. Note that the expression (5.2.2) becomes much simpler if $\omega(x,-x) = 1$, which will be the case in most of the applications.

Exercise 5.2.1. *Check that the above product is associative, and that* $^{*^\omega_\tau}$ *is an involution.*

Remark 5.2.2. *In the corresponding Section 3.4, and more generally in the literature, only the special case* $\tau = \mathbf{0}$ *is considered. We introduced all these isomorphic structures because they help in understanding* τ*-quantizations in pseudodifferential theory.*

Lemma 5.2.3. *For two functions* f *and* g *in* $C_c(G;\mathscr{C})$ *and for* $\tau \in \mathsf{End}(G)$*, the function* $f *^\omega_\tau g$ *belongs to* $C_c(G;\mathscr{C})$*. With the composition law* $*^\omega_\tau$ *and the involution* $^{*^\omega_\tau}$*, the completion* $L^1(G;\mathscr{C})$ *of* $C_c(G;\mathscr{C})$ *with respect to the norm* $\|\cdot\|_1$ *is a* B^**-algebra. These* B^**-algebras are isomorphic for different* τ*'s.*

Proof. The fact that $L^1(G;\mathscr{C})$ is stable under the product $*^\omega_\tau$ follows from the relations

$$\|\theta_{\tau(y-x)}\left[f(y)\right]\theta_{(\mathbf{1}-\tau)y}\left[g(x-y)\right]\theta_{-\tau x}\left[\omega(y,x-y)\right]\| \le \|f(y)\|\,\|g(x-y)\|,$$

and

$$\int_G \|(f *^\omega_\tau g)(x)\|\mathrm{d}x \le \int_G \Big[\int_G \|f(y)\|\,\|g(x-y)\|\mathrm{d}y\Big]\mathrm{d}x = \|f\|_1\|g\|_1.$$

The associativity of this composition law is easily deduced from the 2-cocycle property of ω. All the other requirements also follow by routine computations.

The isomorphisms are the mappings

$$m_{\tau,\tau'} : L^1(G;\mathscr{C}) \to L^1(G;\mathscr{C}), \quad (m_{\tau,\tau'}f)(x) := \theta_{(\tau'-\tau)x}[f(x)], \qquad x \in G.$$

On the first copy of $L^1(G;\mathscr{C})$ one considers the structure defined by τ' and on the second that defined by τ. Note the obvious relations $m_{\tau,\tau'}m_{\tau',\tau''} = m_{\tau,\tau''}$ and $[m_{\tau,\tau'}]^{-1} = m_{\tau',\tau}$ for all $\tau,\tau',\tau'' \in \mathsf{End}(G)$. □

We recall that *a* C^**-norm* on a *-algebra has to satisfy $\|A^*A\| = \|A\|^2$. Since C^*-norms have many technical advantages and since $\|\cdot\|_1$ has not this C^*-property, we shall make now some adjustments, valid in an abstract setting (see Definition 3.4.2 for a simplified version of the following construction). A B^*-algebra $\mathfrak{C}$ with norm $\|\cdot\|$ is called *an* A^**-algebra* when it admits a C^*-norm or, equivalently, when it has an injective representation in a Hilbert space [Tak02, Def. 9.19]. In this case we can consider *the standard* C^**-norm* on it, defined as the supremum of all the C^*-norms, that we shall denote by $|||\cdot|||$. A rather explicit formula for $|||\cdot|||$ is $|||A||| = \sup\{\|\pi(A)\|_{\mathscr{B}(\mathcal{H})} \mid (\mathcal{H},\pi) \text{ is a representation}\}$. One has by Lemma 2.4.14 that $|||A||| \le \|A\|$ for all $A \in \mathfrak{C}$. The completion with respect to this norm will be a C^*-algebra containing $\mathfrak{C}$ as a dense *-subalgebra. We call it *the enveloping* C^**-algebra of* $\mathfrak{C}$. It is known that $\big(L^1(G;\mathscr{C}), *^\omega_\tau, ^{*^\omega_\tau}, \|\cdot\|_1\big)$ is indeed an A^*-algebra[1].

[1] In the general setting of twisted crossed product C^*-algebra, this fact is not trivial. The argument uses the existence of an approximate unit, see [PR89, Rem. 2.6], [BS70, Thm. 3.3] and the Appendix of [PR90]. Fortunately, for our (abelian) twisted C^*-dynamical system, the regular representation induces the necessary injective representation of $L^1(G;\mathscr{C})$, as we shall see in the proof of Proposition 5.4.6.

Definition 5.2.4. The enveloping C^*-algebra of $\big(L^1(G;\mathscr{C}), *^{\omega}_{\tau}, {}^{*^{\omega}_{\tau}}, \|\cdot\|_1\big)$ is called *the twisted crossed product of $\mathscr{C}$ by G associated with the twisted action (θ,ω) and the endomorphism τ*. It will be denoted by $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$.

The C^*-algebra $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ has a rather abstract nature. But most of the time one uses efficiently the fact that $L^1(G;\mathscr{C})$ is a dense *-subalgebra, on which everything is very explicitly defined. Let us even observe that the algebraic tensor product $L^1(G)\odot\mathscr{C}$ may be identified with the dense *-subspace of $L^1(G;\mathscr{C})$ (hence of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ also) formed of functions with finite-dimensional range. The isomorphism $m_{\tau,\tau'}$ extends nicely to an isomorphism from $\mathscr{C} \rtimes^{\omega}_{\theta,\tau'} G$ to $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$.

The next lemma shows clearly the importance of twisted crossed products as a way to bring together the information contained in a twisted dynamical system, see Theorem 3.4.1 for the untwisted version.

Lemma 5.2.5. *Let $(\mathcal{H},\pi,U)$ be a covariant representation of the (abelian) twisted C^*-dynamical system $(\mathscr{C},G,\theta,\omega)$, and let $\tau\in\mathsf{End}(G)$. Then $\pi\rtimes_\tau U$ defined on $L^1(G;\mathscr{C})$ by*

$$(\pi\rtimes_\tau U)f := \int_G \pi\left[\theta_{\tau y}\big(f(y)\big)\right] U_y \,\mathrm{d}y$$

extends to a representation of $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$, called the integrated form of (π,U). *One has $\pi\rtimes_{\tau'} U = (\pi\rtimes_\tau U)\circ m_{\tau,\tau'}$ if $\tau,\tau'\in\mathsf{End}(G)$.*

Proof. Some easy computations show that $\pi\rtimes_\tau U$ is a representation of the B^*-algebra $\big(L^1(G;\mathscr{C}), *^{\omega}_{\tau}, {}^{*^{\omega}_{\tau}}\big)$. Then, by taking into account that $\|(\pi\rtimes_\tau U)f\| \le \|f\|_1$, $\forall f\in L^1(G;\mathscr{C})$, one gets that $\pi\rtimes_\tau U$ extends to $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ by density and, by approximation, this extension has all the required algebraic properties.

The relation $\pi\rtimes_{\tau'} U = (\pi\rtimes_\tau U)\circ m_{\tau,\tau'}$ is checked readily on $L^1(G;\mathscr{C})$ and obviously extends to the full twisted crossed product. □

Let us mention that an analogue of Theorem 3.4.8 also holds in this more general setting. Indeed, one can recover the covariant representation from $\pi\rtimes_\tau U$. Actually, there is a bijective correspondence between covariant representations of a twisted dynamical system and non-degenerate representations of the twisted crossed product. This correspondence preserves equivalence, irreducibility and direct sums. We do not give explicit formulae, since we do not use them.

5.3 Group cohomology

We recall some definitions in group cohomology. They will be used in the next sections to show that standard matters as gauge invariance and τ-quantizations have a cohomological flavour. Now they will serve to isolate twisted dynamical systems for which a generalization of the Schrödinger representation exists.

Let G be an abelian, locally compact group and $\mathscr{U}$ a topological abelian group. Note that in our applications $\mathscr{U}$ will usually not be locally compact, being the unitary group of the multiplier algebra of an abelian C^*-algebra, as in Section 5.1. We also assume that there exists a continuous action θ of G by automorphisms of $\mathscr{U}$. We shall use for G and $\mathscr{U}$ additive and multiplicative notations, respectively.

The class of all continuous functions : $G^n \to \mathscr{U}$ is denoted by $C^n(G;\mathscr{U})$; it is obviously an abelian group (we use once again multiplicative notations). Elements of $C^n(G;\mathscr{U})$ are called *(continuous) n-cochains.* For any $n \in \mathbb{N}$, we define *the coboundary map* $\delta^n : C^n(G;\mathscr{U}) \ni \rho \mapsto \delta^n(\rho) \in C^{n+1}(G;\mathscr{U})$ by

$$\begin{aligned}&[\delta^n(\rho)]\,(x_1,\ldots,x_n,x_{n+1})\\ &:= \theta_{x_1}\,[\rho(x_2,\ldots,x_{n+1})]\prod_{j=1}^{n}\rho(x_1,\ldots,x_j+x_{j+1},\ldots,x_{n+1})^{(-1)^j}\rho(x_1,\ldots,x_n)^{(-1)^{n+1}}.\end{aligned}$$

It is easily shown that δ^n is a group morphism and that $\delta^{n+1}\big(\delta^n(\rho)\big) = 1$ for any $n \in \mathbb{N}$. It follows that $\mathsf{Ran}(\delta^n) \subset \mathsf{Ker}(\delta^{n+1})$.

Definition 5.3.1. *(i)* $Z^n(G;\mathscr{U}) := \mathsf{Ker}(\delta^n)$ *is called* the set of n-cocycles (on G, with coefficients in $\mathscr{U}$)*.*

(ii) $B^n(G;\mathscr{U}) := \mathsf{Ran}(\delta^{n-1})$ *is called* the set of n-coboundaries*.*

Let us note that $Z^n(G;\mathscr{U})$ and $B^n(G;\mathscr{U})$ are subgroups of $C^n(G;\mathscr{U})$, and that $B^n(G;\mathscr{U}) \subset Z^n(G;\mathscr{U})$.

Definition 5.3.2. *The quotient* $H^n(G;\mathscr{U}) := Z^n(G;\mathscr{U})/B^n(G;\mathscr{U})$ *is called* the n'th group of cohomology (of G with coefficients in $\mathscr{U}$)*. Its elements are called* classes of cohomology*.*

In the sequel, we shall need only the cases $n = 0, 1, 2$, which we outline now for convenience. For $n = 0$, parts of the definitions are simple conventions. We set $C^0(G;\mathscr{U}) := \mathscr{U}$. One has $[\delta^0(\varphi)]\,(x) = \theta_x(\varphi)\varphi^{-1}$, for any $\varphi \in \mathscr{U}$, $x \in G$. This implies that $Z^0(G;\mathscr{U}) = \{\varphi \in \mathscr{U} \mid \varphi \text{ is a fixed point}\}$. By convention, $B^0(G;\mathscr{U}) = \{1\}$.

The mapping $\delta^1 : C^1(G;\mathscr{U}) \to C^2(G;\mathscr{U})$ is given by

$$\big[\delta^1(\lambda)\big]\,(x,y) = \lambda(x)\,\theta_x[\lambda(y)]\,\lambda(x+y)^{-1}.$$

Thus a 1-cochain λ is in $Z^1(G;\mathscr{U})$ if it is *a crossed morphism*, *i.e.* if it satisfies $\lambda(x)\theta_x[\lambda(y)] = \lambda(x+y)$ for any $x, y \in G$. Particular cases are the elements of $B^1(G;\mathscr{U})$ (called *principal morphisms*), those of the form $\lambda(x) = \theta_x(\varphi)\varphi^{-1}$ for some $\varphi \in \mathscr{U}$.

For $n = 2$ one encounters a situation which was already taken into account in the definition of twisted dynamical systems. The formula for the coboundary map is

$$\big[\delta^2(\omega)\big]\,(x,y,z) = \theta_x[\omega(y,z)]\,\omega(x+y,z)^{-1}\,\omega(x,y+z)\,\omega(x,y)^{-1}.$$

Thus a 2-cocycle is just a function satisfying the relation (5.1.1). $B^2(G;\mathscr{U})$ is composed of 2-cocycles of the form $\omega(x,y) = \lambda(x)\,\theta_x[\lambda(y)]\,\lambda(x+y)^{-1}$ for some 1-cochain λ.

In the applications, we shall consider for $\mathscr{U}$ the unitary group of an algebra of functions defined on the group G itself. An example of special importance will be the group $\mathscr{U} = C(G;\mathbb{T})$, endowed with the strict topology, which correspond to the unitary group of the multiplier algebra of $C_0(G)$. In this case, the groups of cohomology are particularly simple.

Lemma 5.3.3. *For any locally compact abelian group G and for any $n \geq 1$, one has $H^n\big(G; C(G;\mathbb{T})\big) = \{1\}$.*

Proof. Let $\rho^n \in Z^n\big(G; C(G;\mathbb{T})\big)$, i.e. ρ^n is a continuous n-cochain satisfying for any $y_1, \dots, y_{n+1} \in G$

$$\theta_{y_1}\left[\rho^n(y_2,\dots,y_{n+1})\right] \prod_{j=1}^{n} \rho^n(y_1,\dots,y_j+y_{j+1},\dots,y_{n+1})^{(-1)^j} \rho^n(y_1,\dots,y_n)^{(-1)^{n+1}} = 1.$$

We set in this relation $y_1 = q$, $y_j = x_{j-1}$ for $j \geq 2$ and rephrase it as

$$\begin{aligned}&\theta_q\left[\rho^n(x_1,\dots,x_n)\right]\\ =&\rho^n(q+x_1,x_2,\dots,x_n)\prod_{j=1}^{n-1}\rho^n(q,x_1,\dots,x_j+x_{j+1},\dots,x_n)^{(-1)^j}\rho^n(q,x_1,\dots,x_{n-1})^{(-1)^n},\end{aligned}$$

which is an identity in $C(G;\mathbb{T})$. One calculates both sides at the point $x = 0$ and obtain

$$\begin{aligned}\left[\rho^n(x_1,\dots,x_n)\right](q) &= \left[\rho^n(q+x_1,x_2,\dots,x_n)\right](0)\\ &\cdot\prod_{j=1}^{n-1}\left[\rho^n(q,x_1,\dots,x_j+x_{j+1},\dots,x_n)^{(-1)^j}\right](0)\left[\rho^n(q,x_1,\dots,x_{n-1})^{(-1)^n}\right](0).\end{aligned}$$

This means exactly $\rho^n = \delta^{n-1}(\rho^{n-1})$ for

$$\left[\rho^{n-1}(z_1,\dots,z_{n-1})\right](q) := \left[\rho^n(q,z_1,\dots,z_{n-1})\right](0) \tag{5.3.1}$$

and thus any n-cocyle is at least formally a n-coboundary.

We show now that ρ^{n-1} has the right continuity properties. Let us recall that if $C(G;\mathbb{T})$ is endowed with the topology of uniform convergence on compact sets of G and if Y is a locally compact space, then $C\big(Y; C(G;\mathbb{T})\big)$ can naturally be identified with $C(G \times Y;\mathbb{T})$ (the proof of this statement is an easy exercise). So ρ^n can be interpreted as an element of $C(G \times G^n;\mathbb{T})$. Being obtained from ρ^n by a restriction ρ^{n-1} belongs to $C(G^n;\mathbb{T})$, and thus can be interpreted as an element of $C\big(G^{n-1}; C(G;\mathbb{T})\big) \equiv C^{n-1}\big(G; C(G;\mathbb{T})\big)$, which finishes the proof. □

Let us add one more definition which will play a crucial role in the sequel.

Definition 5.3.4. *Let $\mathscr{U}$ be a topological abelian group endowed with a continuous action θ of G by automorphisms of $\mathscr{U}$, and let $\omega \in Z^2(G;\mathscr{U})$. We say that ω is* pseudo-trivial *if there exists another topological abelian group $\mathscr{U}'$ with a similar action θ' of G such that $\mathscr{U}$ is a subgroup of $\mathscr{U}'$, for each $x \in G$ one has $\theta_x = \theta'_x|_{\mathscr{U}}$, and such that $\omega \in B^2(G;\mathscr{U}')$.*

Thus, to produce pseudo-trivial 2-cocycles, one has to find some $\omega \in B^2(G;\mathscr{U}')$ such that $\omega(x,y) \in \mathscr{U} \subset \mathscr{U}'$ for any $x,y \in G$ and such that $(x,y) \mapsto \omega(x,y) \in \mathscr{U}$ is continuous with respect to the topology of $\mathscr{U}$. This is possible in principle because the product $\lambda(x)\theta_x[\lambda(y)][\lambda(x+y)]^{-1}$ can be better-behaved than any of its factors. The particular choice $[\lambda(z)](q) = [\omega(q,z)](0)$ we made in (5.3.1) will lead later on to the familiar transversal gauge for magnetic systems.

Let us emphasize that most of the time pseudo-triviality cannot be improved to a bona fide triviality. Very often, all the functions λ for which one has $\omega = \delta^1(\lambda)$ do not take all their values in $\mathscr{U}$ or miss the right continuity. We shall outline such a situation in the next section.

5.4 Standard twisted crossed products

When trying to transform the formalism of twisted crossed products into a pseudodifferential theory, one has to face the possible absence of an analogue of the Schrödinger representation and this would lead us too far from the initial motivation. The existence of a generalized Schrödinger representation is assured by the pseudo-triviality of the 2-cocycle, and thus we restrict ourselves to a specific class of twisted dynamical systems. In the same time we also restrict to algebras $\mathscr{C}$ of complex continuous functions on G. This also is not quite compulsory for a pseudodifferential theory, but it leads to a simple implementation of pseudo-triviality (by Lemma 5.3.3) and covers easily the important magnetic case.

We first extend of framework introduced in Assumption 4.3.1.

Definition 5.4.1. *Let G be an locally compact abelian group. We call G-*algebra *a C^*-subalgebra $\mathscr{C}$ of $BC_u(G)$ which is G-invariant,* i.e. *$\theta_x(\varphi) := \varphi(\cdot + x) \in \mathscr{C}$ for any $\varphi \in \mathscr{C}$ and $x \in G$, and which contains $C_0(G)$.*

The C^*-algebra $BC_u(G)$ is the largest one on which the action θ of translations with elements of G is norm-continuous. But we shall denote by θ_x even the x-translation on $C(G)$, the *-algebra of all continuous complex functions on G (which is not a normed algebra if G is not compact). The restriction of θ_x on $BC(G)$ is only strictly continuous.

Note that in the previous definition, the assumption $C_0(G) \subset \mathscr{C}$ implies that G can be identified with a dense subset of the Gelfand spectrum Ω of $\mathscr{C}$. If $\mathscr{C}$ is unital, then Ω is a compactification of G, see the beginning of Section 4.3 for the special case $G = \mathbb{R}^d$.

Now, if $\mathscr{C}$ is a G-algebra, then $(\mathscr{C}, G, \theta)$ is a C^*-dynamical system. If we twist it, we get:

Definition 5.4.2. A standard twisted dynamical system *is an (abelian) twisted C^*-dynamical system $(\mathscr{C}, G, \theta, \omega)$ for which $\mathscr{C}$ is a G-algebra. The C^*-algebra $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ is called* a standard twisted crossed product.

Proposition 5.4.3. *If $(\mathscr{C}, G, \theta, \omega)$ is a standard twisted dynamical system, then ω is pseudo-trivial.*

Note that a slightly more general statement and proof is provided in [MPR05, Prop.2.14]. In our context, if is sufficient to observe that $\mathscr{U}(\mathscr{C})$ can naturally be identified with a subgroup of $C(G;\mathbb{T})$, and that the strict topology on $\mathscr{U}(\mathscr{C})$ is finer than the strict topology of $C(G;\mathbb{T})$. The 2-cocycle ω can hence be considered as an element of $Z^2\big(G;C(G;\mathbb{T})\big)$, which coincides with $B^2\big(G;C(G;\mathbb{T})\big)$ by Lemma 5.3.3, and this proves the statement.

Remark 5.4.4. *If ω,ω' are two cohomologous elements of $Z^2\big(G;\mathscr{U}(\mathscr{C})\big)$,* i.e. $\omega = \delta^1(\lambda)\omega'$ *for some $\lambda \in C^1\big(G;\mathscr{U}(\mathscr{C})\big)$, then the C^*-algebras $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \rtimes^{\omega'}_{\theta,\tau} G$ are naturally isomorphic: on $L^1(G;\mathscr{C})$ the isomorphism is given by $\left[i^\lambda_\tau(f)\right](x) := \theta_{-\tau x}[\lambda(x)]f(x)$. Thus $C_0(G) \rtimes^{\omega}_{\theta,\tau} G$ does not depend on ω but on its class of cohomology; this will be strengthened in Proposition 5.4.6. However this does not work if λ only belongs to $C^1\big(G;C(G;\mathbb{T})\big)$ and $\mathscr{C}$ is not $C_0(G)$; in general $\theta_{-\tau x}[\lambda(x)]f(x)$ gets out of $\mathscr{C}$ and i^λ_τ is no longer well-defined. For ω and ω' defining different classes of cohomology, $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \rtimes^{\omega'}_{\theta,\tau} G$ are in general different C^*-algebras.*

In the sequel we fix a standard twisted dynamical system $(\mathscr{C},G,\theta,\omega)$. One observes that the untwisted system $(\mathscr{C},G,\theta)$ always has an obvious covariant representation $(\mathcal{H},\pi,U)$, with $\mathcal{H} := L^2(G)$ (with the Haar measure), $\pi(\varphi) \equiv \varphi(X) =$ the multiplication operator with φ, and $[U_y u](x) := u(x+y)$. Note that the right action is again considered, as in Example 5.1.4. Let us now choose $\lambda \in C^1\big(G;C(G;\mathbb{T})\big)$ such that $\delta^1(\lambda) = \omega$ (this identity taking place in $Z^2\big(G;C(G;\mathbb{T})\big)$). We set $U^\lambda_y := \pi\big(\lambda(y)\big)U_y$. Explicitly, for any $x \in G$ and $u \in \mathcal{H}$, $\left[U^\lambda_y u\right](x) = [\lambda(y)](x)u(x+y) \equiv \lambda(x;y)u(x+y)$. Let us already mention that the point (ii) in the next proposition is at the root of gauge invariance for magnetic pseudodifferential operators.

Proposition 5.4.5. *(i) $(\mathcal{H},\pi,U^\lambda)$ is a covariant representation of $(\mathscr{C},G,\theta,\omega)$,*

(ii) If μ is another element of $C^1\big(G;C(G;\mathbb{T})\big)$ such that $\delta^1(\mu) = \omega$, then there exists $\varphi \in C(G;\mathbb{T})$ such that $\mu(x) = \theta_x(\varphi)\varphi^{-1}\lambda(x)$, $\forall x \in G$. Moreover, $U^\mu_x = \pi(\varphi^{-1})U^\lambda_x\pi(\varphi)$ for all $x \in G$.

Proof. The proof of the first statement consists in trivial verifications. For the second statement, one first notes that $\mu\lambda^{-1}$ belongs to $\mathsf{Ker}(\delta^1) = Z^1\big(G;C(G;\mathbb{T})\big)$. Since this set is equal to $B^1\big(G;C(G;\mathbb{T})\big)$ by Lemma 5.3.3, there exists $\varphi \in C^0\big(G;C(G;\mathbb{T})\big) \equiv C(G;\mathbb{T})$ satisfying $\mu(x) = \theta_x(\varphi)\varphi^{-1}\lambda(x)$, $\forall x \in G$. The last claim of the proposition follows from $\pi[\theta_x(\varphi)]U_x = U_x\pi(\varphi)$. $\square$

We call $(\mathcal{H},\pi,U^\lambda)$ *the Schrödinger covariant representation associated with the* 1-*cochain* λ. Let us now recall the detailed form of the composition laws on $L^1(G;\mathscr{C})$. For simplicity we shall use notations as $f(x;y)$ for $[f(y)](x)$ and $\omega(x;y,z)$ for $[\omega(y,z)](x)$. With these notations and for any $f,g \in L^1(G;\mathscr{C})$, the relations (5.2.1) and (5.2.2) read respectively

$$(f *^{\omega}_{\tau} g)(q;x) = \int_G f\big(q+\tau(y-x);y\big)\, g\big(q+(\mathbf{1}-\tau)y;x-y\big)\, \omega\big(q-\tau x;y,x-y\big)\, \mathrm{d}y$$

and

$$(f^{*^\omega_\tau})(q;x) = \omega\big(q-\tau x; x, -x\big)^{-1}\overline{f\big(q+(\mathbf{1}-2\tau)x; -x\big)},$$

where x, y, q are elements of G.

Let us also denote for convenience by $\mathfrak{Rep}^\lambda_\tau$ the integrated representation $\pi \rtimes_\tau U^\lambda$ in $L^2(G)$ of the twisted crossed product $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$, see also Lemma 5.2.5. Its explicit action on $f \in L^1(G;\mathscr{C})$ and $u \in L^2(G)$ is given by

$$\begin{aligned}\big[\big(\mathfrak{Rep}^\lambda_\tau(f)\big)\, u\big]\,(x) &= \int_G f(x+\tau y; y)\,\lambda(x;y)\,u(x+y)\,\mathrm{d}y \\ &= \int_G f\big((\mathbf{1}-\tau)x+\tau y; y-x\big)\,\lambda(x;y-x)\,u(y)\,\mathrm{d}y.\end{aligned}$$

We gather some important properties of $\mathfrak{Rep}^\lambda_\tau$ in:

Proposition 5.4.6. *(i) $\mathfrak{Rep}^\lambda_\tau[C_0(G) \rtimes^\omega_{\theta,\tau} G] = \mathscr{K}\big(L^2(G)\big)$, the C^*-algebra of all compact operators in $L^2(G)$.*

(ii) $\mathfrak{Rep}^\lambda_\tau$ is a irreducible and faithful representation of $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$ in $L^2(G)$, for any G-algebra $\mathscr{C}$,

(iii) In the setting of Proposition 5.4.5.(ii), one has $\mathfrak{Rep}^\mu_\tau(f) = \pi(\varphi^{-1})\mathfrak{Rep}^\lambda_\tau(f)\pi(\varphi)$.

Proof. (i) Since $\delta^1(\lambda) = \omega$ in $Z^2\big(G; C(G;\mathbb{T})\big)$, we can then consider the following isomorphism

$$\begin{aligned} i^\lambda_\tau : \Big(L^1\big(G; C_0(G)\big), *^1_{\mathbf{0}}, {}^{*^1_{\mathbf{0}}}\Big) &\to \big(L^1\big(G; C_0(G)\big), *^\omega_\tau, {}^{*^\omega_\tau}\big)\,, \\ \big[i^\lambda_\tau(f)\big]\,(x) &= \theta_{-\tau x}\left[\lambda^{-1}(x)\,f(x)\right], \end{aligned} \tag{5.4.1}$$

that extends to an isomorphism between the non-twisted crossed product $C_0(G) \rtimes^1_{\theta,\mathbf{0}} G$ and the twisted crossed product $C_0(G) \rtimes^\omega_{\theta,\tau} G$ (this is consistent with Remark 5.4.4). One easily checks that $\mathfrak{Rep}^\lambda_\tau\big[i^\lambda_\tau(f)\big] = \int_G \pi[f(x)]U_x\,\mathrm{d}x$ for all f in $\Big(L^1(G;\mathscr{C}), *^1_{\mathbf{0}}, {}^{*^1_{\mathbf{0}}}\Big)$. But it is known that the image of $C_0(G) \rtimes^1_{\theta,\mathbf{0}} G$ through the representation $\pi \rtimes U \equiv \mathfrak{Rep}^1_{\mathbf{0}}$ is equal to the algebra $\mathscr{K}\left(L^2(G)\right)$ of compact operators in $L^2(G)$, *cf.* for example [GI02, Cor. 4.1].

(ii) Since $C_0(G) \subset \mathscr{C}$, then $C_0(G) \rtimes^\omega_{\theta,\tau} G$ can be identified to a C^*-subalgebra of $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$ and the irreducibility of $\mathfrak{Rep}^\lambda_\tau\left(\mathscr{C} \rtimes^\omega_{\theta,\tau} G\right)$ follows from the irreducibility of $\mathscr{K}\left(L^2(G)\right)$, by (i).

Let us now recall that the regular representation of the twisted dynamical system $(\mathscr{C}, G, \theta, \omega)$ has been introduced in Example 5.1.4. In particular, we can choose in this representation the Hilbert space $L^2(G)$ and the representation π of $\mathscr{C}$ by operators of multiplication. One thus obtained the representation $\big(L^2\big(G; L^2(G)\big), \tilde{\pi}, \tilde{U}\big)$, with the maps $\tilde{\pi}$ and $\tilde{U}$ defined in (5.1.5). Since $L^2\big(G; L^2(G)\big)$ is canonically isomorphic to $L^2(G \times G)$, let us set $\xi(\cdot; x) := \xi(x)$ and introduce the unitary operator

$W^\lambda : L^2(G\times G) \to L^2(G\times G)$, $[W^\lambda \xi](x;y) := \lambda(x;y)\,\xi(x;x+y)$. Its adjoint is given by $[(W^\lambda)^*\xi](x;y) = \lambda^{-1}(x;y-x)\,\xi(x;y-x)$. Some easy computations show then that $\big[(W^\lambda)^*\,\tilde\pi(\varphi)\,W^\lambda\xi\big]\,(x;y) = \varphi(y)\,\xi(x;y)$. Moreover, one has

$$\begin{aligned}\Big[(W^\lambda)^*\,\tilde U_z\,W^\lambda\,\xi\Big]\,(x;y) &= \lambda^{-1}(x;y-x)\,\omega(x;y-x,z)\,\lambda(x;y-x+z)\,\xi(x;y+z)\\ &= \lambda(y;z)\,\xi(x;y+z),\end{aligned}$$

where we have used that $\omega = \delta^1(\lambda)$. Equivalently, one has $(W^\lambda)^*\,\tilde\pi(\varphi)\,W^\lambda = \mathbf{1}\otimes\varphi(X)$ and $(W^\lambda)^*\,U_z\,W^\lambda = \mathbf{1}\otimes\lambda(X;z)U_z \equiv \mathbf{1}\otimes U^\lambda_z$ in $L^2(G)\otimes L^2(G)$. Thus the regular representation is unitarily equivalent to the representation $(L^2(G)\otimes L^2(G), \mathbf{1}\otimes\pi, \mathbf{1}\otimes U^\lambda)$. Since the regular representation induces a faithful representation $\tilde\pi\rtimes U$ of $\mathscr{C}\rtimes^\omega_{\theta,\mathbf{0}} G$ in $L^2\big(G;L^2(G)\big)$, *cf.* Theorem 3.11 of [PR89], the Schrödinger representation induces a faithful representation of $\mathscr{C}\rtimes^\omega_{\theta,\tau} G$ in $L^2(G)$ for any $\tau\in\mathsf{End}(G)$.

(iii) The proof of this statement consists in a simple verification. □

Exercise 5.4.7. *Check that the map i^λ_τ introduced in the previous proof defines an isomorphism between the B^*-algebras $\Big(L^1\big(G;C_0(G)\big),*^1_{\mathbf{0}},{}^{*^1_{\mathbf{0}}}\Big)$ and $\big(L^1\big(G;C_0(G)\big),*^\omega_\tau,{}^{*^\omega_\tau}\big)$.*

Chapter 6

Pseudodifferential calculus

The aim of this chapter is to stress the link between the algebraic framework introduced so far, and the usual pseudodifferential calculus. The first section is related to the content of Chapter 4, and is based on [MPR05, Sec. 1.1]. It consists mainly in an introduction to the Weyl calculus and to the corresponding Moyal product. The subsequent sections are slightly more general and closely related to Chapter 5. The arguments are borrowed from [MPR05, Sec. 3], and the mentioned link is clearly established.

6.1 The Weyl calculus

In section 4.1 we have seen how to define multiplication operators $\varphi(X)$ and convolution operators $\varphi(D)$ on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$. A natural question is how to define a more general operator $f(X, D)$ on $L^2(\mathbb{R}^d)$ for a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$.

This can be seen as the problem of constructing a functional calculus $f \mapsto f(X, D)$ for the family $X_1, \dots, X_d, D_1, \dots, D_d$ of $2d$ self-adjoint, non-commuting operators. One also would like to define a multiplication $(f, g) \mapsto f \circ g$ satisfying $(f \circ g)(X, D) = f(X, D)g(X, D)$ as well as an involution $f \to f^\circ$ leading to $f^\circ(X, D) = f(X, D)^*$. The deviation of $\circ$ from pointwise multiplication is imputable to the fact that X and D do not commute.

The solution of these problems is called *the Weyl calculus*, or simply *the pseudodifferential calculus*. In order to define it, let us set $\Xi := \mathbb{R}^d \times \hat{\mathbb{R}}^d$, which corresponds to the direct product of a locally compact abelian group G and of its dual group $\hat{G}$. Elements of Ξ will be denoted by $\mathsf{x} = (x, \xi)$, $\mathsf{y} = (y, \eta)$ and $\mathsf{z} = (z, \zeta)$. We also set

$$\sigma(\mathsf{x}, \mathsf{y}) := \sigma\big((x, \xi), (y, \eta)\big) = y \cdot \xi - x \cdot \eta$$

for the standard *symplectic form* on Ξ. The prescription for $f(X, D) \equiv \mathfrak{Op}(f)$ with $f : \Xi \to \mathbb{C}$ is then defined for $u \in \mathcal{H}$ and $x \in \mathbb{R}^d$ by

$$[\mathfrak{Op}(f)u](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} f\left(\frac{x+y}{2}, \eta\right) u(y)\, \mathrm{d}y\, \mathrm{d}\eta, \qquad (6.1.1)$$

the involution is $f^\circ(\mathsf{x}) := \overline{f(\mathsf{x})}$ and the multiplication (called *the Moyal product*) is

$$(f \circ g)(\mathsf{x}) := \frac{4^d}{(2\pi)^{2d}} \int_\Xi \int_\Xi e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})} f(\mathsf{y})\, g(\mathsf{z})\, \mathrm{d}\mathsf{y}\, \mathrm{d}\mathsf{z}. \tag{6.1.2}$$

Obviously, these formulas must be taken with some care: for many symbols f and g they need a suitable reinterpretation. Also, the normalization factors should always be checked once again, since they mainly depend on the conventions of each author.

Exercise 6.1.1. *Check that if $f(x,\xi) = f(\xi)$ (f is independent of x), then $\mathfrak{Op}(f) = f(D)$, while if $f(x,\xi) = f(x)$ (f is independent of ξ), then $\mathfrak{Op}(f) = f(X)$.*

Beside the encouraging results contained in the previous exercise, let us try to show where all the above formulas come from. We consider the strongly continuous unitary maps $\mathbb{R}^d \ni x \mapsto U_x \in \mathscr{U}(\mathcal{H})$ and $\hat{\mathbb{R}}^d \ni \xi \mapsto V_\xi := e^{-iX\cdot\xi} \in \mathscr{U}(\mathcal{H})$, acting on $\mathcal{H}$ as

$$[U_x u](y) = u(y+x) \quad \text{and} \quad [V_\xi u](y) = e^{-iy\cdot\xi}\, u(y), \qquad u \in \mathcal{H},\ y \in \mathbb{R}^d.$$

These operators satisfy the *Weyl form of the canonical commutation relations*

$$U_x V_\xi = e^{-ix\cdot\xi}\, V_\xi U_x, \qquad x \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d, \tag{6.1.3}$$

as well as the identities $U_x U_{x'} = U_{x'} U_x$ and $V_\xi V_{\xi'} = V_{\xi'} V_\xi$ for $x, x' \in \mathbb{R}^d$ and $\xi, \xi' \in \hat{\mathbb{R}}^d$. These can be considered as a reformulation of the content of Exercise 4.1.3 in terms of bounded operators.

A convenient way to condense the maps U and V in a single one is to define *the Schrödinger Weyl system* $\{W(x,\xi) \mid x \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d\}$ by

$$W(\mathsf{x}) \equiv W(x,\xi) := e^{\frac{i}{2}x\cdot\xi}\, U_x V_\xi = e^{-\frac{i}{2}x\cdot\xi}\, V_\xi U_x, \tag{6.1.4}$$

which satisfies the relation $W(\mathsf{x})W(\mathsf{y}) = e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\, W(\mathsf{x}+\mathsf{y})$ for any $\mathsf{x}, \mathsf{y} \in \Xi$. This equality encodes all the commutation relations between the basic operators X and D. Explicitly, the action of W on $u \in \mathcal{H}$ is given by

$$[W(x,\xi)u](y) = e^{-i\left(\frac{1}{2}x+y\right)\cdot\xi}\, u(y+x), \qquad x, y \in \mathbb{R}^d,\ \xi \in \hat{\mathbb{R}}^d. \tag{6.1.5}$$

Now, recall that for a family of m commuting self-adjoint operators $S_1, \dots, S_m$ one usually defines a functional calculus by the formula $f(S) := \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \check{f}(t) e^{-it\cdot S} \mathrm{d}t$, where $t \cdot S = t_1 S_1 + \ldots + t_m S_m$ and $\check{f}$ is the inverse Fourier transform of f, see Remark 1.7.13 for a simplified version of this equality. The formula (6.1.1) can be obtained by a similar computation. For that purpose, let us define the *symplectic Fourier transformation* $\mathcal{F}_\Xi :\ \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ by

$$(\mathcal{F}_\Xi f)(\mathsf{x}) := \frac{1}{(2\pi)^d} \int_\Xi e^{i\sigma(\mathsf{x},\mathsf{y})} f(\mathsf{y})\, \mathrm{d}\mathsf{y}.$$

Now, for any function $f : \Xi \to \mathbb{C}$ belonging to the Schwartz space $\mathcal{S}(\Xi)$, we set

$$\mathfrak{Op}(f) := \frac{1}{(2\pi)^d} \int_\Xi (\mathcal{F}_\Xi^{-1} f)(\mathsf{x}) W(\mathsf{x}) \, d\mathsf{x}. \tag{6.1.6}$$

By using (6.1.5), one gets formula (6.1.1). Then it is easy to verify that the relation $\mathfrak{Op}(f)\mathfrak{Op}(g) = \mathfrak{Op}(f \circ g)$ holds for $f, g \in \mathcal{S}(\Xi)$ if one uses the Moyal product introduced in (6.1.2).

Exercise 6.1.2. *Check that the above statements are correct, and in particular that the normalization factors are suitably chosen.*

6.2 Generalized pseudodifferential algebras

We have introduced in Section 5.4 the standard twisted crossed products $(\mathscr{C}, G, \theta, \omega)$, as well as their family of Schrödinger representations $(\mathcal{H}, \pi, U^\lambda)$ with $\mathcal{H} = L^2(G)$, defined by pseudo-trivializations of the 2-cocycle ω. We shall now observe that by a partial Fourier transformation, we get from these data a sort of pseudodifferential calculus. More precisely, certain classes of functions on $G \times \hat{G}$ will be organised in C^*-algebras with a natural involution and a product involving ω and generalizing the Moyal product introduced in (6.1.2). The composition between the partial Fourier transformation and the Schrödinger representation will lead to a rule of assigning operators to symbols belonging to these C^*-algebras.

Let us consider the locally compact abelian group G and its dual group $\hat{G}$ endowed with normalized Haar measures in such a way that the Fourier transformations

$$\mathcal{F}_G : L^1(G) \to C_0(\hat{G}), \quad (\mathcal{F}_G b)(\xi) = \int_G \overline{\xi(x)}\, b(x) \, dx$$

and

$$\overline{\mathcal{F}}_G : L^1(G) \to C_0(\hat{G}), \quad \big(\overline{\mathcal{F}}_G b\big)(\xi) = \int_G \xi(x)\, b(x) \, dx$$

induce unitary maps from $L^2(G)$ to $L^2(\hat{G})$. The inverses of these maps act on $L^2(\hat{G}) \cap L^1(\hat{G})$ as $\big(\overline{\mathcal{F}}_{\hat{G}} c\big)(x) = \int_{\hat{G}} \xi(x) c(\xi) \, d\xi$ and $(\mathcal{F}_{\hat{G}} c)(x) = \int_{\hat{G}} \overline{\xi(x)}\, c(\xi) \, d\xi$.

Let us now consider the standard twisted C^*-dynamical system $(\mathscr{C}, G, \theta, \omega)$. We define the mapping $\mathbf{1} \otimes \overline{\mathcal{F}}_G : L^1(G; \mathscr{C}) \to C_0(\hat{G}; \mathscr{C})$ by $\big[\big(\mathbf{1} \otimes \overline{\mathcal{F}}_G\big)(f)\big](\xi) = \int_G \xi(x) f(x) \, dx$ (equality in $\mathscr{C}$). We recall that $L^1(G) \odot \mathscr{C}$ is a dense subspace of $L^1(G; \mathscr{C})$ and observe that $\big(\mathbf{1} \otimes \overline{\mathcal{F}}_G\big)(a \otimes b) = a \otimes \big(\overline{\mathcal{F}}_G b\big)$. Let us now also fix an element $\tau \in \mathsf{End}(G)$. We transport all the structure of the Banach *-algebra $(L^1(G; \mathscr{C}), *^\omega_\tau, ^{*^\omega_\tau}, \|\cdot\|_1)$ to the corresponding subset of $C_0(\hat{G}; \mathscr{C})$ via $\mathbf{1} \otimes \overline{\mathcal{F}}_G$. The space $\big(\mathbf{1} \otimes \overline{\mathcal{F}}_G\big) L^1(G; \mathscr{C})$ will also be a Banach *-algebra with a composition law $\circ^\omega_\tau$, an involution $^{\circ^\omega_\tau}$ and the norm $\|(\mathbf{1} \otimes \overline{\mathcal{F}}_G^{-1}) \cdot \|_1$. Its enveloping C^*-algebra will be denoted by $\mathfrak{C}^\omega_{\mathscr{C},\tau}$. The map $\mathbf{1} \otimes \overline{\mathcal{F}}_G$ extends canonically to an isomorphism between $\mathscr{C} \rtimes^\omega_{\theta,\tau} G$ and $\mathfrak{C}^\omega_{\mathscr{C},\tau}$. We remark that $\big(\mathbf{1} \otimes \overline{\mathcal{F}}_G\big)[L^1(G) \odot \mathscr{C}]$ is already not very explicit, since one has no direct characterization of the space $\overline{\mathcal{F}}_G[L^1(G)]$.

Concerning $\mathfrak{C}^{\omega}_{\mathscr{C},\tau}$, we do not even know if it consists entirely of $\mathscr{C}$-valued distributions on $\hat{G}$ (whenever this makes sense). However, usually one can work efficiently on suitable dense subsets.

We deduce now the explicit form of the composition law and of the involution. Let us simply denote $\mathbf{1}\otimes\overline{\mathcal{F}_G}$ by $\mathfrak{F}$. One gets for any $f,g\in\mathfrak{F}L^1(G;\mathscr{C})$ (be careful with the position of the arguments)

$$\begin{aligned}(f\circ^{\omega}_{\tau} g)(x;\xi) &:= \left(\mathfrak{F}\left[(\mathfrak{F}^{-1}f)*^{\omega}_{\tau}(\mathfrak{F}^{-1}g)\right]\right)(x;\xi)\\ &= \int_G\int_G\int_{\hat{G}}\int_{\hat{G}}\xi(y)\,\overline{\eta(z)}\,\overline{\zeta(y-z)}\;f(x+\tau(z-y);\eta)\cdot\\ &\qquad\cdot\; g(x+(\mathbf{1}-\tau)z;\zeta)\;\omega(x-\tau y;z,y-z)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\eta\,\mathrm{d}\zeta\end{aligned}$$

and

$$\begin{aligned}\left(f^{\circ^{\omega}_{\tau}}\right)(x;\xi) &:= \left(\mathfrak{F}\left[\left(\mathfrak{F}^{-1}f\right)^{*^{\omega}_{\tau}}\right]\right)(x;\xi)\\ &= \int_G\int_{\hat{G}}\left[\xi\cdot\eta^{-1}\right](y)\;\omega(x-\tau y;y,-y)^{-1}\;\overline{f(x+(1-2\tau)y;\eta)}\,\mathrm{d}y\,\mathrm{d}\eta.\end{aligned}$$

Both expressions make sense as iterated integrals; under more stringent conditions on f and g, the integrals will be absolutely convergent.

Exercise 6.2.1. *Show that in the special case $\tau=\frac{1}{2}\mathbf{1}$ and $\omega=1$, the above formulas correspond to the Moyal product $\circ$ and to the involution $^{\circ}$ introduced in Section 6.1.*

The constructions and formulae presented above can be given (with some slight adaptations) for any (abelian) twisted dynamical system. However, since we are considering a standard twisted dynamical system, it means that ω is pseudo-trivial. Thus, for any continuous function $\lambda: G\to C(G;\mathbb{T})$ such that $\delta^1(\lambda)=\omega$, the corresponding Schrödinger covariant representation $(\mathcal{H},\pi,U^{\lambda})$ gives rise to the Schrödinger representation of $\mathscr{C}\rtimes^{\omega}_{\theta,\tau}G$ that we have denoted by $\mathfrak{Rep}^{\lambda}_{\tau}$ in the previous chapter. As a consequence, we get a representation of $\mathfrak{C}^{\omega}_{\mathscr{C},\tau}$ just by composing with $\mathfrak{F}^{-1}$; and this representation will be denoted by $\mathfrak{Op}^{\lambda}_{\tau}$. By simple computations one obtains:

Proposition 6.2.2. *(i) The representation $\mathfrak{Op}^{\lambda}_{\tau}:=\mathfrak{Rep}^{\lambda}_{\tau}\circ\mathfrak{F}^{-1}:\mathfrak{C}^{\omega}_{\mathscr{C},\tau}\to\mathscr{B}(\mathcal{H})$ is faithful and acts on $f\in\mathfrak{F}L^1(G;\mathscr{C})$ with $u\in\mathcal{H}$ and $x\in G$ by the formula*

$$\left[\mathfrak{Op}^{\lambda}_{\tau}(f)u\right](x)=\int_G\int_{\hat{G}}\eta(x-y)\,\lambda(x;y-x)\,f\big((1-\tau)x+\tau y;\eta\big)\,u(y)\,\mathrm{d}y\,\mathrm{d}\eta \quad (6.2.1)$$

where the right-hand side is viewed as an iterated integral.

(ii) If $\mu\in C^1\big(G;C(G;\mathbb{T})\big)$ is another 1-cochain, giving a second pseudo-trivialization of the 2-cocycle ω, then $\mu=\delta^0(c)\lambda$ for some $c\in C(G;\mathbb{T})$ and $\mathfrak{Op}^{\lambda}_{\tau}$, $\mathfrak{Op}^{\mu}_{\tau}$ are unitarily equivalent:

$$\pi(c^{-1})\,\mathfrak{Op}^{\lambda}_{\tau}(f)\,\pi(c)=\mathfrak{Op}^{\mu}_{\tau}(f),\quad \forall f\in\mathfrak{C}^{\omega}_{\mathscr{C},\tau}. \quad (6.2.2)$$

Remark 6.2.3. *One can again observe that in the special case* $\tau = \frac{1}{2}\mathbf{1}$ *and with the choice* $\lambda = 1$ *(absence of* 2*-cocycle), the formula provided in* (6.2.1) *corresponds to the expression provided in* (6.1.1)*.*

Let us recall from from Section 5.2 that for different τ's, the C^*-algebras $\mathscr{C} \rtimes^{\omega}_{\theta,\tau} G$ and $\mathscr{C} \times^{\omega}_{\theta,\tau} G$ are isomorphic, and therefore $\mathfrak{C}^{\omega}_{\theta,\tau}$ and $\mathfrak{C}^{\omega}_{\theta,\tau}$ are also isomorphic. More precisely, recall that $(m_{\tau,\tau'} f)(q;x) = f(q + (\tau' - \tau)x; x)$ for any $x, q \in G$ and $f \in L^1(G;\mathcal{A})$. Note that this isomorphism satisfies then $\mathfrak{Op}^{\lambda}_{\tau'} = \mathfrak{Op}^{\lambda}_{\tau} \circ m_{\tau,\tau'}$ (here $\circ$ is simply the composition) and thus gives the transformation of the τ-symbol of a generalized pseudodifferential operator into its τ'-symbol.

As already mentioned, in the general literature on twisted crossed product C^*-algebras only the special case $\tau = \mathbf{0}$ is considered. However, in order to make the connection with the usual Weyl calculus on the group $\mathbb{R}^d$, the special choice $\tau = \frac{1}{2}\mathbf{1}$ had to be considered, and this is the reason why we have introduced the larger family $\tau \in \mathsf{End}(G)$. We now support the assertion that the choice of the parameter τ is in fact a matter of ordering. Indeed, let us assume that the G-algebra $\mathscr{C}$ is unital, see Definition 5.4.1 for the notion of G-algebra. Then any element $f = 1 \otimes b$ is in $\mathfrak{C}^{\omega}_{\theta,\tau}$ for any $b : \hat{G} \to \mathbb{C}$ with $\mathcal{F}_{\hat{G}} b \in L^1(G)$. In addition, the operator $\mathfrak{Op}^{\lambda}_{\tau}(1 \otimes b)$ does not depend on τ, see formula (6.2.1). We denote it by $\mathfrak{op}^{\lambda}(b)$; its action on $u \in \mathcal{H}$ is given by

$$\left[\mathfrak{op}^{\lambda}(b)u\right](x) = \int_G \lambda(x; y-x)\,[\overline{\mathcal{F}}_{\hat{G}} b](y-x)\, u(y)\,\mathrm{d}y.$$

Finally, by considering then arbitrary element $a \in \mathscr{C}$, simple computations for $\tau = \mathbf{0}$ and $\tau = \mathbf{1}$ show that $\mathfrak{Op}^{\lambda}_{\mathbf{0}}(a \otimes b) = \pi(a)\mathfrak{op}^{\lambda}(b)$ and $\mathfrak{Op}^{\lambda}_{\mathbf{1}}(a \otimes b) = \mathfrak{op}^{\lambda}(b)\pi(a)$, where $\pi(a)$ denotes the multiplication operator by the function a.

Extension 6.2.4. *Let us stress once more that the set of functions for which the above integrals are absolutely convergent can be rather small, and certainly too small for various applications. Several possible extensions are then possible. A first approach would be to deal with multiplier algebras, as sketched in [MPR05, Sec. 3.3]. An approach by duality (but in a less general framework) has been introduced in [MP04]. Alternatively, technics involving oscillatory integrals have been discussed in [LMR10], also in the magnetic framework introduced in the following chapter. All these extensions allow us to consider the expressions introduced in this chapter for a much larger class of symbols.*

Chapter 7

Magnetic systems

In this chapter we shall see how all the previous constructions can be used when a magnetic field is considered on $\mathbb{R}^d$.

Very briefly, a *continuous magnetic field* is described by a closed continuous 2-form B defined on $\mathbb{R}^d$. It is well-known that any such field B may be written as the differential $\mathrm{d}A$ of a 1-forms A called a *vector potential*, which is highly non-unique (the gauge ambiguity). By using coordinates, one has

$$B_{jk} = \partial_j A_k - \partial_k A_j \qquad \text{for any } j,k \in \{1,\dots \mathrm{d}\}.$$

In the presence of the field $B = \mathrm{d}A$, the prescription (6.1.1) has to be modified. This topic was very rarely touched in the literature and the following wrong solution appears: The *minimal coupling principle* says roughly that the momentum D should be replaced with *the magnetic momentum* $\Pi^A = D - A(X)$. This originated in Lagrangian classical mechanics and works well also at the quantum level as long as we consider operators which are polynomials of order less or equal to 2. But if one just replaces in (6.1.1) the expression $f\big((x+y)/2,\eta\big)$ by $f\big((x+y)/2,\eta - A(x+y)/2\big)$ one gets a formula which misses the right gauge covariance. Indeed, let us denote the result of this procedure for some function f in phase space by $\mathfrak{Op}_A(f)$. If another vector potential A' is chosen such that $A' = A + \nabla\rho$ with ρ a scalar function, then $\mathrm{d}A' = \mathrm{d}A$. But the expected formula $\mathfrak{Op}_{A'}(f) = e^{i\rho}\mathfrak{Op}_A(f)e^{-i\rho}$ is verified for some simple cases (A, A' linear and f arbitrary, or f polynomial of order strictly less than 3 in η and A, A' arbitrary), but it fails in general.

Thus, the aim of the following sections is two show that the correct solution can directly be inferred from the formalism constructed before, without the invocation of a minimal coupling principle. The content of this chapter is borrowed from the three references [MPR05, MPR07, LMR10].

7.1 Magnetic twisted dynamical systems

From now on, the group G will always be $\mathbb{R}^d$, with its usual action θ by translations. The 2-cocycle will be defined in terms of the magnetic field. More precisely, the magnetic field on $\mathbb{R}^d$ is a closed continuous 2-form B. Since on $\mathbb{R}^d$ we have canonical global coordinates, we shall speak freely of the components B_{jk} of B; they are continuous real functions on $\mathbb{R}^d$ satisfying $B_{kj} = -B_{jk}$ and (in the distributional sense)

$$\partial_j B_{kl} + \partial_l B_{jk} + \partial_k B_{lj} = 0 \qquad \forall j, k, l \in \{1, \dots, d\}.$$

It is well-known that $B = \mathrm{d}A$ for some 1-form A on $\mathbb{R}^d$, called a vector potential, which is highly non-unique. For simplicity, we shall consider only continuous A; this is always possible since at least one continuous vector potential always exists, namely *the transversal gauge* which is defined by

$$A_j(x) := -\sum_{k=1}^{d} \int_0^1 B_{jk}(sx) s x_k \, \mathrm{d}s. \tag{7.1.1}$$

Given a k-form C on $\mathbb{R}^d$ and a compact k-surface $\gamma \subset \mathbb{R}^d$, we define

$$\Gamma^C(\gamma) := \int_\gamma C,$$

this integral having a well-defined parametrization independent meaning. We shall mainly encounter circulations of 1-forms along linear segments $\gamma = [x, y]$ and fluxes of 2-forms through triangles $\gamma = \langle x, y, z \rangle$. In particular, for a continuous magnetic field B one defines

$$\omega^B(q; x, y) := e^{-i\Gamma^B(\langle q, q+x, q+x+y \rangle)} \qquad \text{for all } x, y, q \in \mathbb{R}^d. \tag{7.1.2}$$

From now on, let us fix a $\mathbb{R}^d$-algebra $\mathscr{C}$, *i.e.* a C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is invariant under the actions of $\mathbb{R}^d$ by translations. Note that in Definition 5.4.1 we have also assumed that $C_0(\mathbb{R}^d) \subset \mathscr{C}$, but that this additional condition is not necessary here. By Gelfand representation, we know that $\mathscr{C} \cong C_0(\Omega)$, with Ω the spectrum of $\mathscr{C}$. In this setting, the additional assumption $C_0(\mathbb{R}^d) \subset \mathscr{C}$ allowed one to identify $\mathbb{R}^d$ with a dense subset of Ω. Let us now consider $C(\Omega)$, the set of continuous functions on Ω. If $\mathscr{C}$ is not unital, then such functions can be unbounded. The simplest example is obtained by considering $\mathscr{C} = C_0(\mathbb{R}^d)$ with Ω equal to $\mathbb{R}^d$. Taking this observation into account, let us now define a magnetic field which is related to the $\mathbb{R}^d$-algebra $\mathscr{C}$:

Definition 7.1.1. *A magnetic field B is of type $\mathscr{C}$ with $\mathscr{C} \cong C_0(\Omega)$ if all its components $\{B_{jk}\}_{j,k=1}^d$ belong to $C(\Omega; \mathbb{R})$.*

Clearly, if $B_{jk} \in \mathscr{C}$ for any $j, k \in \{1, \dots, d\}$, then B is a magnetic field of type $\mathscr{C}$. However, the previous definition is more general, and unbounded magnetic field can be considered in this setting. We recall that the notion of standard twisted system has been introduced in Definition 5.4.2.

Lemma 7.1.2. *If B is a magnetic field of type $\mathscr{C}$, then $(\mathscr{C}, \mathbb{R}^d, \theta, \omega^B)$ is a standard twisted dynamical system.*

Proof. The proof that ω^B is a normalized 2-cocyle, *i.e.* that it satisfies relations (5.1.1) and (5.1.2), follows easily by direct computations (for the first one use the Stokes Theorem for the closed 2-form B and the tetrahedron of vertices $q, q+x, q+x+y, q+x+y+z$).

We now show that ω^B has the right continuity properties. It should define a mapping

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \to \left[\omega^B(x,y)\right](\cdot) \equiv \omega^B(\cdot;x,y) \in C(\Omega;\mathbb{T}), \tag{7.1.3}$$

continuous with respect to the topology of uniform convergence on compact subsets of Ω. But this is equivalent to the fact that ω^B defines un element of $C(\Omega\times\mathbb{R}^d\times\mathbb{R}^d;\mathbb{T})$. Note that this type of statement already appeared in the proof of Lemma 5.3.3. Taking into account obvious properties of the exponential, this amounts to the fact that the function

$$\varphi^B : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad \varphi^B(q;x,y) := \Gamma^B(\langle q, q+x, q+x+y\rangle)$$

can be viewed as a continuous function on $\Omega\times\mathbb{R}^d\times\mathbb{R}^d$.

We use the parametrization

$$\varphi^B(q;x,y) = \sum_{j,k=1}^{d} x_j\, y_k \int_0^1 \int_0^1 s\, B_{jk}(q+sx+sty)\,\mathrm{d}s\,\mathrm{d}t.$$

Since the continuous action θ on $\mathscr{C}$ defines a continuous mapping θ on Ω, one has the continuous correspondence $\Omega\times\mathbb{R}^d\times\mathbb{R}^d \ni (q;x,y) \to q+sx+sty = \theta_{sx+sty}(q) \in \Omega$. Since B_{jk} is seen as a continuous function from Ω to $\mathbb{R}$, the assertion follows easily. □

Exercise 7.1.3. *Work out the details of the previous proof, and in particular show that ω^B satisfies the two conditions* (5.1.1) *and* (5.1.2)*.*

From now on, we can call $(\mathscr{C}, \mathbb{R}^d, \theta, \omega^B)$ *the twisted dynamical system associated with the abelian algebra $\mathscr{C}$ and the magnetic field B*. In most of the cases the 2-cocycle $\omega^B \in Z^2\big(\mathbb{R}^d;\mathscr{U}(\mathscr{C})\big)$ is not trivial. But as Proposition 5.4.3 shows, it is pseudo-trivial. In fact, its pseudo-trivialization can be achieved by a vector potential. Any continuous 1-form A defines a 1-cochain $\lambda^A \in C^1\big(\mathbb{R}^d; C(\mathbb{R}^d;\mathbb{T})\big)$ via its circulation:

$$\left[\lambda^A(x)\right](q) \equiv \lambda^A(q;x) = e^{-i\Gamma^A([q,q+x])} = e^{-ix\cdot\int_0^1 A(q+sx)\,\mathrm{d}s}. \tag{7.1.4}$$

As soon as $\mathrm{d}A = B$, we have $\delta^1(\lambda^A) = \omega^B$ $\big($a priori with respect to $C(\mathbb{R}^d;\mathbb{T})\big)$, by a suitable version of Stokes Lemma. As said above, the transversal gauge offers a continuous vector potential corresponding to a given B. Actually, this is consistent with the choice (5.3.1) of a pseudo-trivialization of ω^B: for $q,x\in\mathbb{R}^d$, $\lambda(q;x) := \omega^B(0;q,x) = e^{-i\Gamma^B(\langle 0,q,q+x\rangle)}$ and it follows immediately that $\Gamma^B(\langle 0,q,q+x\rangle) = \Gamma^A([q,q+x])$, with A given by (7.1.1).

Since specific standard twisted dynamical systems can be constructed based on any magnetic field of type $\mathscr{C}$, the whole formalism of the preceding chapters is available. In particular, twisted crossed product algebra $\mathscr{C} \rtimes_{\theta,\tau}^{\omega^B} \mathbb{R}^d$, also denoted by $\mathscr{C} \rtimes_{\theta,\tau}^{B} \mathbb{R}^d$ and their Schrödinger representations are at hand. Note that as always, the dependence on τ is within isomorphism, and that for any continuous B the C^*-algebra $C_0(\mathbb{R}^d) \rtimes_{\theta,\tau}^{B} \mathbb{R}^d$ is isomorphic to $\mathscr{K}(\mathcal{H})$, the ideal of all compact operators in $\mathcal{H} = L^2(\mathbb{R}^d)$.

Let us close this section with some comments on the magnetic momentum, already introduced in the preamble of this chapter. The fact that the magnetic 2-cocycle ω^B satisfies

$$\omega^B(q; sx, tx) = 1, \quad \forall q, x \in \mathbb{R}^d \quad \text{and} \quad \forall s, t \in \mathbb{R} \tag{7.1.5}$$

leads directly to the magnetic momenta. Indeed, let us fix some continuous A such that $\mathrm{d}A = B$, and thus $\delta^1(\lambda^A) = \omega^B$. Then λ^A satisfies for all $q, x \in \mathbb{R}^d$ and all $s, t \in \mathbb{R}$: $\lambda^A(q; sx+tx) = \lambda^A(q; sx)\lambda^A(q+sx; tx)$ (note that in general, if λ is not the exponential of a circulation this will not be true). We consider then the Schrödinger covariant representation $(\mathcal{H}, \pi, U^A)$ with $\mathcal{H} = L^2(\mathbb{R}^d)$, $\pi(a) = a(X)$ and $U^A = U^{\lambda^A}$ defined by

$$[U^A_y u](x) \equiv [U^A(y)u](x) = \lambda^A(x; y)u(x+y), \quad x, y \in \mathbb{R}^d, \quad u \in \mathcal{H}.$$

The unitary operators $\{U^A(y)\}_{y\in\mathbb{R}^d}$ are called *the magnetic translations.* They often appear in the physical literature. One has, by a short computation,

$$U^A(sx + tx) = U^A(sx)U^A(tx), \quad \forall x \in \mathbb{R}^d, \quad \forall s, t \in \mathbb{R} \tag{7.1.6}$$

and this also implies $U^A(-x) = U^A(x)^{-1} = U^A(x)^*$ for all $x \in \mathbb{R}^d$. In fact, the formula

$$U^A(y)U^A(z) = \pi[\omega^B(y, z)]U^A(y + z), \quad y, z \in \mathbb{R}^d$$

shows that (7.1.6) is equivalent with (7.1.5). For $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, let us set $U^A_t(x) := U^A(tx)$. By (7.1.6), we observe that $\{U^A_t(x)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group in $\mathcal{H}$ for any x. Thus, by Stone Theorem (see Theorem 1.7.12), it has a self-adjoint generator that moreover depends linearly (as a linear operator on $\mathcal{H}$) on the vector $x \in \mathbb{R}^d$. Thus we denote it by $x\cdot\Pi^A$ and call it *the projection on x of the magnetic momentum associated with the vector potential A*. For any index $j \in \{1, ..., n\}$ we set $\Pi^A_j := e_j \cdot \Pi^A$ the projection of the magnetic momentum on the j'th vector of the canonical base in $\mathbb{R}^d$. A direct computation shows that on $C^\infty_c(\mathbb{R}^d)$ one has $\Pi^A_j = D_j - A_j(X)$.

7.2 Magnetic pseudodifferential calculus

In this section, we adapt the results presented in Section 6.1 when a magnetic field is also present. Most of the following formulas appeared already in the more general setting of Section 6.2, but this section can be seen as a useful résumé for the interested reader.

Let us directly start by introducing the analog of the Weyl system recalled in (6.1.4) but in the presence of a magnetic field. For the time being, B is any continuous magnetic field on $\mathbb{R}^d$ and A is any corresponding continuous vector potential. Associated with the Schrödinger covariant representation $(\mathcal{H}, \pi, U^A)$ defined above, we can now define *the magnetic Weyl system* W^A by

$$\Xi \ni \mathsf{x} \mapsto W^A\big(\mathsf{x}\big) := e^{-\frac{i}{2}x\cdot\xi} V_\xi U^A(x) \in \mathscr{U}(\mathcal{H}).$$

These unitary operators satisfy then the relations

$$W^A(\mathsf{x})\, W^A(\mathsf{y}) = e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\, \pi[\omega^B(x,y)]\, W^A(\mathsf{x}+\mathsf{y})$$

for any $\mathsf{x} = (x, \xi)$ and $\mathsf{y} = (y, \eta)$.

Exercise 7.2.1. *Check the above relations*

For any $f \in \mathcal{S}(\Xi)$ we can then write explicitly the operator $\mathfrak{Op}^A(f) := \mathfrak{Op}^{\lambda^A}_{1/2}(f)$ in $\mathcal{H}$ which has been introduced in Proposition 6.2.2, namely

$$\big[\mathfrak{Op}^A(f)u\big]\,(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} e^{-i\Gamma^A([x,y])} f\left(\frac{x+y}{2}, \eta\right) u(y)\, \mathrm{d}y\, \mathrm{d}\eta.$$

Note that this formula can be called *the magnetic Weyl calculus*. Furthermore, it is easily observed that this is an integral operator with kernel

$$K^A := \tilde{\lambda}^A\, S^{-1}\,(\mathbf{1} \otimes \overline{\mathcal{F}}_{\hat{\mathbb{R}}^d})$$

where $\tilde{\lambda}^A(x, y) := \lambda^A(x; y-x)$ and $\big(S^{-1}h\big)(x, y) = h\left(\frac{x+y}{2}, x-y\right)$. With this formula, we can now extend the map K^A and thus define $\mathfrak{Op}^A(F)$ for any $F \in \mathcal{S}'(\Xi)$ as the integral operator with kernel $K^A(F)$, defined on $\mathcal{S}(\mathbb{R}^d)$ with values in $\mathcal{S}'(\mathbb{R}^d)$. It seems legitimate to view the correspondence $f \to \mathfrak{Op}^A(f)$ as a functional calculus for the family of self-adjoint operators $X_1, \ldots, X_d, \Pi^A_1, \ldots, \Pi^A_d$. The high degree of non-commutativity of these $2d$ operators stays at the origin of the sophistication of the symbolic calculus. The commutation relations

$$i[X_j, X_k] = 0, \quad i[\Pi^A_j, X_k] = \delta_{jk}, \quad i[\Pi^A_j, \Pi^A_k] = -B_{jk}(X), \quad j, k = 1, \ldots, d \qquad (7.2.1)$$

collapse for $B = 0$ to the canonical commutation relations satisfied by X and D, see Exercise 4.1.3. But they are much more complicated, especially when B is not a polynomial. The main mathematical miracle that allows, however, a nice treatment is the fact that (7.2.1) can be recast in the form of a covariant representation of a twisted dynamical system.

Let us stress once more that the functional calculus that we have defined is gauge covariant, in the sense that it satisfies the property: If $A' = A + \nabla\varphi$ with $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous, then $\mathfrak{Op}^{A'}(f) = e^{i\varphi(X)}\mathfrak{Op}^A(f)e^{-i\varphi(X)}$. This gauge covariance property may be seen as a special instance of Proposition 6.2.2.

The extension of the usual Moyal product has a particular form in the magnetic setting. More precisely, by adapting the formula obtained in Section 6.2 to the magnetic 2-cocyle and for $\tau = 1/2$, one obtains on $\mathcal{S}(\Xi)$ the composition and the involution:

$$(f \circ^B g)(\mathsf{x}) = \frac{4^d}{(2\pi)^d}\int_\Xi\int_\Xi e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})}e^{-i\Gamma^B(\langle x-z+y,y-x+z,z-y+x\rangle)}f(\mathsf{y})g(\mathsf{z})\,\mathrm{d}\mathsf{y}\,\mathrm{d}\mathsf{z}, \quad (7.2.2)$$

with $\mathsf{x},\mathsf{y},\mathsf{z} \in \Xi$, and

$$f^{\circ^B}(\mathsf{x}) = \overline{f(\mathsf{x})}, \quad \forall \mathsf{x} \in \Xi.$$

Note that with these formulas, one has $(f \circ^B g)^{\circ^B} = g^{\circ^B} \circ^B f^{\circ^B}$ as well as

$$\mathfrak{Op}^A(f \circ^B g) = \mathfrak{Op}^A(f)\mathfrak{Op}^A(g), \qquad \text{and} \qquad \mathfrak{Op}^A(f^{\circ^B}) = \mathfrak{Op}^A(f)^*.$$

Exercise 7.2.2. *Without relying on the content of the previous sections, check directly these equalities.*

We remark that the involution $^{\circ^B}$ and the product $\circ^B$ are defined intrinsically, without any choice of a vector potential. The choice is only needed when we represent the resulting structures on the Hilbert space $L^2(\mathbb{R}^d)$. We call (7.2.2) *the magnetic Moyal product*. The involution $^{\circ^B}$ does not depend on B at all. This is no longer true if $\tau \neq 1/2$. The property $\omega^B(x, -x) = 1$, $\forall x \in \mathbb{R}^d$, is also used to get the simple form of $^{\circ^B}$.

Let us now assume that B is of type $\mathscr{C}$ for some $\mathbb{R}^d$-algebra $\mathscr{C}$. The C^*-algebra $\mathfrak{C}^{\omega^B}_{\mathscr{C},1/2}$, introduced in Section 6.2, will be denoted by $\mathfrak{C}^B_{\mathscr{C}}$. We call it *the C^*-algebra of pseudodifferential symbols of class $\mathscr{C}$ associated with B*. We recall that it is essentially a partial Fourier transform of the twisted crossed product $\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d$. The formulas defining the magnetic Weyl calculus make sense at least on the dense subset $(\mathbf{1} \otimes \overline{\mathcal{F}}_{\mathbb{R}^d})L^1(\mathbb{R}^d;\mathscr{C})$, with iterated integrals. The extension of $\mathfrak{Op}^A$ is a faithful representation of the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$ for any continuous A with $\mathrm{d}A = B$. If $C_0(\mathbb{R}^d) \subset \mathscr{C}$, then $\mathfrak{Op}^A$ is irreducible.

We close this section with some arguments about one possible extension for the product $\circ^B$. Indeed, as already mentioned in the Extension 6.2.4, the integrals defining $f \circ^B g$ are absolutely convergent only for restricted classes of symbols. In order to deal with more general distributions, an extension by duality was proposed in [MP04] under an additional smoothness condition on the magnetic field. So let us assume that the components of the magnetic field are $C^\infty_{pol}(\mathbb{R}^d)$-functions, *i.e.* they are indefinitely derivable and each derivative is polynomially bounded. The duality approach is based on the observation [MP04, Lem. 14] : For any f, g in the Schwartz space $\mathcal{S}(\Xi)$, we have

$f \circ^B g \in \mathcal{S}(\Xi)$, and

$$\int_\Xi [f \circ^B g](\mathsf{x})\mathrm{d}\mathsf{x} = \int_\Xi [g \circ^B f](\mathsf{x})\mathrm{d}\mathsf{x} = \int_\Xi f(\mathsf{x})\, g(\mathsf{x})\mathrm{d}\mathsf{x} = \langle f, \overline{g}\rangle =: (f, g).$$

As a consequence, by using the associativity of $\circ^B$ and the symmetry of $(\cdot,\cdot)$, one easily deduces that for $f, g, h \in \mathcal{S}(\Xi)$, one has

$$(f \circ^B g, h) = (f, g \circ^B h) = (g, h \circ^B f).$$

Definition 7.2.3. *For any distribution $F \in \mathcal{S}'(\Xi)$ and any function $f, h \in \mathcal{S}(\Xi)$ we define*

$$(F \circ^B f, h) := (F, f \circ^B h), \qquad (f \circ^B F, h) := (F, h \circ^B f)$$

The expressions $F \circ^B f$ and $f \circ^B F$ are *a priori* tempered distributions. The Moyal algebra is precisely the set of elements of $\mathcal{S}'(\Xi)$ that preserves regularity by composition.

Definition 7.2.4. The magnetic Moyal algebra $\mathscr{M}(\Xi)$ *is defined by*

$$\mathscr{M}(\Xi) := \left\{F \in \mathcal{S}'(\Xi) \mid F \circ^B f \in \mathcal{S}(\Xi) \text{ and } f \circ^B F \in \mathcal{S}(\Xi) \text{ for all } f \in \mathcal{S}(\Xi)\right\}.$$

For two distributions F and G in $\mathscr{M}(\Xi)$, the magnetic Moyal product *can be extended by*

$$(F \circ^B G, h) := (F, G \circ^B h) \quad \text{for all } h \in \mathcal{S}(\Xi).$$

Clearly, the set $\mathscr{M}(\Xi)$ with this composition law and the complex conjugation $F \mapsto F^\circ$ is a unital *-algebra. An important result [MP04, Prop. 23] concerning the Moyal algebra is that it contains $C^\infty_{pol,u}(\Xi)$, the space of infinitely derivable complex functions on Ξ having uniform polynomial growth at infinity. Finally let us quote a result linking $\mathscr{M}(\Xi)$ with the functional calculus $\mathfrak{Op}^A$ [MP04, Prop. 21] : For any vector potential A belonging to $C^\infty_{pol}(\mathbb{R}^d)$, $\mathfrak{Op}^A$ is an isomorphism of *-algebras between $\mathscr{M}(\Xi)$ and $\mathscr{B}[\mathcal{S}(\mathbb{R}^d)] \cap \mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$, where $\mathscr{B}[\mathcal{S}(\mathbb{R}^d)]$ and $\mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$ are, respectively, the spaces of linear continuous operators on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

Remark 7.2.5. *The extension by duality also gives compositions $\mathscr{M}(\Xi) \circ^B \mathcal{S}'(\Xi) \subset \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \circ^B \mathscr{M}(\Xi) \subset \mathcal{S}'(\Xi)$. One checks plainly that associativity holds for any three factors product with two factors belonging to $\mathscr{M}(\Xi)$ and one in $\mathcal{S}'(\Xi)$.*

7.3 Magnetic Schrödinger operators

From now on, we consider for simplicity a $\mathbb{R}^d$-algebra $\mathscr{C}$ which is unital and which contains $C_0(\mathbb{R}^d)$. As a consequence, $\mathscr{C} \cong C(\Omega)$ with Ω a compactification of $\mathbb{R}^d$. Then, given a magnetic field B of type $\mathscr{C}$, *cf.* Definition 7.1.1, a continuous vector potential A that generates B and a suitable symbol $h : \hat{\mathbb{R}}^d \to \mathbb{R}$, our aim is to show that the magnetic Schrödinger operator $h(\Pi^A)$ (which needs to be carefully defined) defines an observable affiliated to the C^*-algebra

$$\mathfrak{Op}^A(\mathfrak{C}^B_{\mathscr{C}}) = \mathfrak{Rep}^A\big(\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d\big) \equiv \mathfrak{Rep}^{\lambda A}_{1/2}\big(\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d\big) \subset \mathscr{B}(\mathcal{H}),$$

see Definition 4.3.7 for the precision notion of affiliation. The proof of such a statement is rather difficult and we shall do it under some smoothness conditions on the magnetic field B and on the symbol h. We point out that we prove in fact in Theorem 7.3.2 a stronger result that does not depend on the choice of any particular vector potential.

Definition 7.3.1. *(i) For $s \in \mathbb{R}$, a function $h \in C^\infty(\hat{\mathbb{R}}^d)$ is* a symbol of type s, *written $h \in S^s(\hat{\mathbb{R}}^d)$, if the following condition is satisfied:*

$$\forall \alpha \in \mathbb{N}^d,\ \exists c_\alpha > 0 \ \ \textit{such that}\ \ |(\partial^\alpha h)(\xi)| \leq c_\alpha \langle \xi \rangle^{s-|\alpha|} \ \textit{for all}\ \ \xi \in \hat{\mathbb{R}}^d.$$

(ii) The symbol h is called elliptic *if there exist $R > 0$ and $c > 0$ such that*

$$c\langle \xi \rangle^s \leq h(\xi) \ \ \textit{for all}\ \xi \in \hat{\mathbb{R}}^d \ \ \textit{and}\ \ |\xi| \geq R.$$

We denote by $S^s_{el}(\hat{\mathbb{R}}^d)$ the family of elliptic symbols of type s, and set $S^\infty_{el}(\hat{\mathbb{R}}^d) := \cup_s S^s_{el}(\hat{\mathbb{R}}^d)$. Note that all the classes $S^s(\hat{\mathbb{R}}^d)$ are naturally contained in $C^\infty_{pol,u}(\Xi)$, thus in $\mathscr{M}(\Xi)$. For any $z \in \mathbb{C} \setminus \mathbb{R}$, we also set $r_z : \mathbb{R} \to \mathbb{C}$ by $r_z(\cdot) := (\cdot - z)^{-1}$.

We are in a position to state the main results about affiliation. The proofs of these statements are postponed until the next section.

Theorem 7.3.2. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Then each real $h \in S^\infty_{el}(\hat{\mathbb{R}}^d)$ defines an observable Φ^B_h affiliated to $\mathfrak{C}^B_{\mathscr{C}}$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$ one has*

$$(h - z) \circ^B \Phi^B_h(r_z) = 1 = \Phi^B_h(r_z) \circ^B (h - z). \tag{7.3.1}$$

In fact one even has $\Phi^B_h(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathcal{S}'(\Xi)$, so the compositions can be interpreted as $\mathscr{M}(\Xi) \times \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \times \mathscr{M}(\Xi) \to \mathcal{S}'(\Xi)$.

We shall now consider a scalar potential $V \in \mathscr{C}$. As seen in Theorem 3.4.5 the algebra $\mathscr{C}$ can be identified with part of the multiplier algebra of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Then, a straightforward reformulation of the perturbative argument presented in [ABG96, p. 365–366] allows one to define the observable $\Phi^B_{h,V} := \Phi^B_h + V$. Considering now $h + V \in \mathcal{S}'(\Xi)$ we remark that we can compute the Moyal product

$$(h + V - z) \circ^B \Phi^B_{h,V}(r_z) = (h - z) \circ^B \Phi^B_{h,V}(r_z) + V \circ^B \Phi^B_{h,V}(r_z) = 1$$

by using the explicit formula of $\Phi^B_{h,V}$ given in [ABG96, p. 366]. This leads then to the following statement:

Corollary 7.3.3. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Let also V be a real function in $\mathscr{C}$. Then $\Phi^B_{h,V}$ is an observable affiliated to $\mathfrak{C}^B_{\mathscr{C}}$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$ one has*

$$(h + V - z) \circ^B \Phi^B_{h,V}(r_z) = 1 = \Phi^B_{h,V}(r_z) \circ^B (h + V - z).$$

These statements are elegant, being abstract, but in applications one also needs the represented version:

Corollary 7.3.4. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$, and let V be a real function in $\mathscr{C}$. Let A be a continuous vector potential that generates B. Then $\mathfrak{Op}^A(h) + V(X)$ defines a self-adjoint operator in $\mathcal{H}$ with domain given by the image of the operator $\mathfrak{Op}^A\left[(h-z)^{-1}\right]$ (which do not depend on $z \in \mathbb{C} \setminus \mathbb{R}$). This operator is affiliated to $\mathfrak{Op}^A(\mathfrak{C}^B_{\mathscr{C}})$.*

We finally give a description of the essential spectrum of the observables affiliated to the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$. For the generalized magnetic Schrödinger operators of Theorem 7.3.2, this is expressed in terms of the spectra of so-called *asymptotic operators*. The affiliation criterion and the algebraic formalism introduced above play an essential role in the proof of this result. Note that we shall mimic the approach already used in Section 4.5 in the absence of a magnetic field, and freely use the notations and concepts introduced there.

Recall that $\mathscr{C} \cong C(\Omega)$ with Ω a compactification of $\mathbb{R}^d$. Then, for any $\tau \in \Omega \setminus \mathbb{R}^d$, one sets $\mathcal{O}_\tau$ for the orbit generated by τ, and $\mathcal{Q}_\tau$ for the corresponding quasi-orbit. In this setting, for any $f \in C(\Omega)$, the function $x \mapsto f\big(\theta_x(\tau)\big)$ is an element of $BC_u(\mathbb{R}^d)$, see Exercise 4.5.2 for details. In particular, this construction holds for V and B_{jk} if both belong to $\mathscr{C}$.

Theorem 7.3.5. *Let B be a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$ and let $V \in \mathscr{C}$ be a real function. Assume that $\{\mathcal{Q}_{\tau_i}\}_i$ is a covering of $\partial\Omega$ by quasi-orbits. Then for each real $h \in S^\infty_{el}(\hat{\mathbb{R}}^d)$ one has*

$$\sigma_{ess}\big[\mathfrak{Op}^A(h) + V(X)\big] = \overline{\cup_i \sigma\big[\mathfrak{Op}^{A_i}(h) + V_i(X)\big]}, \tag{7.3.2}$$

where A, A_i are continuous vector potentials for B, $B_i \equiv B|_{\mathcal{Q}_{\tau_i}}$, and $V_i \equiv V|_{\mathcal{Q}_{\tau_i}}$.

Clearly, the computation of the essential spectrum is first performed at an abstract level, *i.e.* without using any representation. This computation is more simple since no vector potentials are involved. Only for convenience and tradition, the previous represented version is also stated. Note also that the proof of this theorem is similar to the one presented in Section 4.5, the 2-cocycles fitting very well with the functoriality of the crossed products. We do not give any details here and refer to [MPR07, Sec. 3] for the interested reader.

7.4 Affiliation in the magnetic case

In this section we provide the proofs of Theorem 7.3.2 and of Corollary 7.3.4. Some technical arguments are postponed to the end of the section. Throughout the section, we assume tacitly all the assumptions of Theorem 7.3.2.

The proof of Theorem 7.3.2 will be based on the following strategy: Let $\mathscr{M}$ be an associative algebra with a composition law denoted by $\circ$ and let $\mathfrak{h}$ be an element of $\mathscr{M}$. Our aim is to find the inverse for $\mathfrak{h}$. Assume that $\mathfrak{h}'$ is another element such that $\mathfrak{h} \circ \mathfrak{h}'$ and $\mathfrak{h}' \circ \mathfrak{h}$ are invertible. These inverses are written $(\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ and $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)}$ respectively. Then, the element $\mathfrak{h}' \circ (\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ is obviously a right inverse for $\mathfrak{h}$ and the element $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)} \circ \mathfrak{h}'$ a left inverse for $\mathfrak{h}$. Both expressions are thus equal to $\mathfrak{h}^{(-1)}$.

In the sequel, we shall take for $\mathfrak{h}$ the strictly positive symbol $h + a$, with a large enough, and for $\mathfrak{h}'$ its pointwise inverse $(h + a)^{-1}$. Finding an inverse $(h + a)^{(-1)}$ for $h + a$ with respect to the composition law $\circ^B$ will lead rather easily to an observable. In the calculations below we shall use tacitly the some approximation procedures. For several arguments we will be forced to get out of the algebra $\mathscr{M} = \mathscr{M}(\Xi)$. This will be easily dealt with by a suitable use of elements of $\mathcal{S}'(\Xi)$.

Note finally that for simplicity, elements of $\hat{\mathbb{R}}^d$ will be denoted by p, k or l.

Proof of Theorem 7.3.2. (i) Let us consider an elliptic symbol h of order s and fix some real number $a \geq -\inf h + 1$. We set $h_a := h + a$, and denote by h_a^{-1} its inverse with respect to pointwise multiplication, *i.e.* $h_a^{-1}(p) := (h(p) + a)^{-1}$ for all $p \in \hat{\mathbb{R}}^d$. It is clear that h_a^{-1} is a symbol of type $-s$. Since both functions h_a and h_a^{-1} belong to $C^\infty_{pol,u}(\Xi)$, and thus to the Moyal algebra $\mathscr{M}(\Xi)$, one can calculate their product. By using (7.2.2) we obtain

$$\big(h_a \circ^B h_a^{-1}\big)(q,p) = \frac{4^d}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\hat{\mathbb{R}}^d} \mathrm{d}k \int_{\mathbb{R}^d} \mathrm{d}y \int_{\hat{\mathbb{R}}^d} \mathrm{d}l \, e^{-2i(k\cdot y - l\cdot x)} \gamma^B(q; 2x, 2y) \frac{h_a(p-k)}{h_a(p-l)}, \tag{7.4.1}$$

with

$$\gamma^B(q; 2x, 2y) := \omega^B\big(q - x - y; 2x, 2(y - x)\big). \tag{7.4.2}$$

The last factor in the integral does not depend on x and y; it can be developed:

$$\frac{h_a(p-k)}{h_a(p-l)} = 1 + \sum_{j=1}^d (l_j - k_j) \frac{\int_0^1 \mathrm{d}t \, (\partial_j h)\big(p - l + t(l-k)\big)}{h(p-l) + a} =: 1 + \sum_{j=1}^d F_{a,j}(p; k, l)\ . \tag{7.4.3}$$

Moreover, let

$$\tilde{\gamma}^B(q; k, l) \equiv (\mathbb{F}\gamma^B)(q; k, l) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \, e^{-ik\cdot y} \, e^{il\cdot x} \, \gamma^B(q; x, y).$$

Then the following equality holds (in the sense of distributions):

$$\int_{\hat{\mathbb{R}}^d} \mathrm{d}k \int_{\hat{\mathbb{R}}^d} \mathrm{d}l \, \tilde{\gamma}^B(q; k, l) = \gamma^B(q; 0, 0) = 1. \tag{7.4.4}$$

Thus, by inserting (7.4.3) and (7.4.4) into (7.4.1), we obtain

$$h_a \circ^B h_a^{-1} = 1 + \sum_{j=1}^{d} f_{a,j},$$

with

$$f_{a,j}(q;p) := \int_{\hat{\mathbb{R}}^d} \mathrm{d}k \int_{\hat{\mathbb{R}}^d} \mathrm{d}l \; \tilde{\gamma}^B(q;k,l)\, F_{a,j}(p;k,l) = \big\langle (\mathbb{F}\gamma^B)(q;\cdot,\cdot), F_{a,j}(p;\cdot,\cdot)\big\rangle. \quad (7.4.5)$$

The last notation is used in order to emphasize the duality between $C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$ and its dual. Indeed, for q, p fixed, Lemma 7.4.2 proves that $F_{a,j}(p;\cdot,\cdot) \in C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$, and Lemma 7.4.1 proves that $\gamma^B(q;\cdot,\cdot) \in C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d)$, from which one infers that $(\mathbb{F}\gamma^B)(q;\cdot,\cdot) \in \big[C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)\big]'$, see [Sch73, Chap. VII, Thm. XV].

(ii) We are now going to deduce some useful estimates on $f_{a,j}$. We set $\langle D_x\rangle \equiv \langle -i\partial_x\rangle$. For α, j fixed and m, n integers that we shall choose below, one has

$$\begin{aligned} |(\partial_p^\alpha f_{a,j})(q;p)| \le &\sup_{x,y\in\mathbb{R}^d} |\langle x\rangle^{-n}\langle y\rangle^{-n}\langle D_x\rangle^m\langle D_y\rangle^m \gamma^B(q;x,y)| \cdot \\ &\big\|\langle x\rangle^{-d}\,\langle y\rangle^{-d}\big\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \big\|\langle D_k\rangle^{n+d}\langle D_l\rangle^{n+d}\langle k\rangle^{-m}\langle l\rangle^{-m}\, \big(\partial_p^\alpha F_{a,j}\big)\,(p;\cdot,\cdot)\big\|_{L^2(\hat{\mathbb{R}}^d\times\hat{\mathbb{R}}^d)} . \end{aligned} \quad (7.4.6)$$

By taking into account (7.4.11), and by some simple computations, one can fix m such that the last factor of (7.4.6) is dominated by $c_n\, a^{-1/\mu}\,\langle p\rangle^{s/\mu-1-|\alpha|}$, with $\mu > \max\{1,s\}$. Then, by using Lemma 7.4.1, one can choose n (depending on m) such that the first factor on the r.h.s. term of (7.4.6) is bounded. Altogether, one obtains

$$|(\partial_p^\alpha f_{a,j})(q;p)| \le c\, a^{-1/\mu}\,\langle p\rangle^{s/\mu-1-|\alpha|}, \quad (7.4.7)$$

where c depends on α and j but not on p, q or a.

(iii) Let us now show that for each j, $\mathfrak{F}^{-1}(f_{a,j})$ is an element of $L^1(\mathbb{R}^d;\mathscr{C})$, and thus belongs to the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$.

By taking into account Lemma 7.4.1, the r.h.s. of the equation (7.4.5) can be rewritten as $\big\langle \gamma^B(q;\cdot,\cdot), (\mathbb{F}^* F_{a,j})(p,\cdot,\cdot)\big\rangle$, in which the duality between $C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\big(C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d)\big)' = \mathbb{F}^* C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$ is emphasized. As γ^B defines a function from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathscr{C}$ (see Lemma 7.4.1) that is of class $C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d)$, we can easily prove that $f_{a,j}(\cdot;p)$ belongs to $\mathscr{C}$, for all $p \in \hat{\mathbb{R}}^d$ (by using partitions of unity on $\mathbb{R}^d \times \mathbb{R}^d$ and by approximating the duality pairing with finite linear combinations of elements in $\mathscr{C}$).

This observation together with (7.4.7) imply that the hypotheses of Lemma 7.4.4 are fulfilled for each $f_{a,j}$, with $t = -(1 - s/\mu) < 0$. It follows that $\mathfrak{F}^{-1}(f_{a,j})$ belongs to $L^1(\mathbb{R}^d;\mathscr{C})$ and that there exists $C > 0$ such that

$$\|\mathfrak{F}^{-1}(f_{a,j})\|_1 \le C\, a^{-1/\mu}.$$

Thus, for a large enough, the strict inequality $\left\| \sum_{j=1}^{d} \mathfrak{F}^{-1}(f_{a,j})\right\|_1 < 1$ holds. It follows that $\mathfrak{F}^{-1}(1+\sum_{j=1}^{d} f_{a,j})$ is invertible in $\widetilde{L^1}$, the minimal unitization of $L^1(\mathbb{R}^d;\mathscr{C})$. Equivalently, $h_a \circ^B h_a^{-1} \equiv 1 + \sum_{j=1}^{d} f_{a,j}$ is invertible in $\widetilde{\mathfrak{F}(L^1)}$, the minimal unitization of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Its inverse will be denoted by $\big(h_a \circ^B h_a^{-1}\big)^{(-1)}$.

(iv) We recall that $h_a^{-1} \in S^{-s}(\hat{\mathbb{R}}^d)$. Then, by Lemma 7.4.4 we get that $h_a^{-1} \in \mathfrak{F}\big(L^1(\mathbb{R}^d)\big) \subset \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Thus $h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)}$ is a well defined element of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Moreover, one readily gets $h_a \circ^B [h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)}] = 1$. For this, just think of h_a and h_a^{-1} as elements of the Moyal algebra $\mathscr{M}(\Xi)$ and interpret $(h_a \circ^B h_a^{-1})^{(-1)} \in \widetilde{\mathfrak{F}(L^1)}$ as an element of $\mathcal{S}'(\Xi)$. The needed associativity follows easily from the definition by duality of the composition law as stated in Remark 7.2.5. In the same way one obtains $[(h_a^{-1} \circ^B h_a)^{(-1)} \circ^B h_a^{-1}] \circ^B h_a = 1$ in $\mathscr{M}(\Xi)$. In conclusion, there exists $a_0 \geq -\inf h + 1$ such that for any $a > a_0$ the symbol h_a possess an inverse with respect to the Moyal product

$$h_a^{(-1)} := h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)} = (h_a^{-1} \circ^B h_a)^{(-1)} \circ^B h_a^{-1} \in \mathcal{S}'(\Xi)$$

that also belongs to $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathfrak{C}^B_{\mathscr{C}}$. The second equality follows from Remark 7.2.5 by straightforward arguments.

(v) We define $\Phi_h^B(r_x) := h_{-x}^{(-1)}$ for $x < -a_0$. Then $\Phi_h^B(r_x) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathfrak{C}^B_{\mathscr{C}} \cap \mathcal{S}'(\Xi)$, its norm is uniformly bounded for x in the given domain and $(h-x) \circ^B \Phi_h^B(r_x) = \Phi_h^B(r_x) \circ^B (h-x) = 1$, as shown above. This allows us to obtain an extension to the half-strip $\{z = x+iy \mid x < -a_0, |y| < \delta\}$ for some $\delta > 0$ by setting

$$\Phi_h^B(r_z) := \Phi_h^B(r_x) \circ^B \{1 + (x-z)\Phi_h^B(r_x)\}^{(-1)}. \tag{7.4.8}$$

It follows that

$$(h-z) \circ^B \Phi_h^B(r_z) = \{(h-x) \circ^B \Phi_h^B(r_x) + (x-z)\Phi_h^B(r_x)\} \circ^B \{1 + (x-z)\Phi_h^B(r_x)\}^{(-1)} = 1.$$

We now prove that the map

$$\{z = x+iy \mid x < -a_0, |y| < \delta\} \ni z \mapsto \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$$

satisfies the resolvent equation. Let us choose two complex numbers z and z' in this domain and subtract the two equations

$$(h-z) \circ^B \Phi_h^B(r_z) = 1, \qquad (h-z') \circ^B \Phi_h^B(r_{z'}) = 1 \tag{7.4.9}$$

in order to get $(h-z) \circ^B \{\Phi_h^B(r_z) - \Phi_h^B(r_{z'})\} + (z'-z)\Phi_h^B(r_{z'}) = 0$. By multiplying at the left with $\Phi_h^B(r_z)$ and by using the associativity, we obtain the resolvent equation

$$\Phi_h^B(r_z) - \Phi_h^B(r_{z'}) = (z-z')\Phi_h^B(r_z) \circ^B \Phi_h^B(r_{z'}).$$

Now, setting $z' = \overline{z} = x - iy$ with $y > 0$ and taking norms we get

$$\|\Phi_h^B(r_z)\|_{\mathfrak{F}(L^1(\mathbb{R}^d;\mathscr{C}))} \leq y^{-1}.$$

With this estimate and formula (7.4.8), the function $z \mapsto \Phi_h^B(r_z)$ can be extended to the domain $\mathbb{C} \setminus [-a_0, +\infty)$, preserving the relations (7.4.9). The resolvent equation may be proved in a similar way to hold on the entire domain $\mathbb{C} \setminus [-a_0, +\infty)$ and analyticity of the defined function follows in an evident way.

(vi) Thus we have got an analytic map $\mathbb{C}\setminus[-a_0, +\infty) \ni z \to \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d; \mathscr{C})\big)$ satisfying the resolvent equation and the symmetry condition. A general argument presented in [ABG96, p. 364] allows now to extend in a unique way the map Φ_h^B to a C^*-algebra morphism $C_0(\mathbb{R}) \to \mathfrak{C}_{\mathscr{C}}^B$. □

We can now provide the represented version of our affiliation criterion.

Proof of Corollary 7.3.4. We shall first consider the case $V = 0$ and then add V as a bounded perturbation.

Let us denote by D_z the range of the operator $\mathfrak{Op}^A[\Phi_h^B(r_z)] \in \mathscr{B}(\mathcal{H})$. By the resolvent identity it follows immediately that it is a subspace of $\mathcal{H}$ that does not depend on $z \in \mathbb{C} \setminus \mathbb{R}$. Thus we set $\mathsf{D}_z \equiv \mathsf{D}$. Since $h \in \mathscr{M}(\Xi)$, one has $\mathfrak{Op}^A(h) \in \mathscr{B}[\mathcal{S}(\mathbb{R}^d)] \cap \mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$. We interpret it as a linear operator in $\mathcal{S}'(\mathbb{R}^d)$ and set $H(A, 0) := \mathfrak{Op}^A(h)|_{\mathsf{D}}$.

Now, by applying $\mathfrak{Op}^A$ to (7.3.1) we get

$$\{H(A,0) - z\mathbf{1}\}\mathfrak{Op}^A[\Phi_h^B(r_z)] = \mathbf{1}$$

and

$$\mathfrak{Op}^A[\Phi_h^B(r_z)]\{\mathfrak{Op}^A(h) - z\mathbf{1}_{\mathcal{S}(\mathbb{R}^d)}\} = \mathbf{1}_{\mathcal{S}(\mathbb{R}^d)}.$$

The first identity shows that $H(A,0)\mathsf{D} \subset \mathcal{H}$. Straightforwardly it is hermitian. The second equality implies that $\mathcal{S}(\mathbb{R}^d) \subset \mathsf{D}$ and thus D is dense in $\mathcal{H}$. By the first equality above the ranges of $H(A,0) \pm i$ both coincide with $\mathcal{H}$. Thus, by a fundamental criterion of self-adjointness, $H(A,0)$ is self-adjoint.

By construction, $\{\mathfrak{Op}^A[\Phi_h^B(r_z)] \mid z \in \mathbb{C}\setminus\mathbb{R}\}$ is the resolvent family of $H(A,0)$, which is therefore affiliated to $\mathfrak{Op}^A\big(\mathfrak{C}_{\mathscr{C}}^B\big)$.

Then we define the standard operator sum $H(A,V) := H(A,0) + V : \mathsf{D} \to \mathcal{H}$. Using the second resolvent equation and the Neumann series the conclusion of the Corollary follows easily using [MPR05, Prop. 2.6]. A different proof could start from the result of Corollary 7.3.3. □

We can now present several technical lemmas which have already been used in the previous proofs.

Lemma 7.4.1. *Assume that the components of the magnetic field B belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Then γ^B, defined in (7.4.2), belongs to $C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d; \mathscr{C})$, or more precisely:*

(a) *for each $x, y \in \mathbb{R}^d$, $\gamma^B(\cdot; x, y) \in \mathscr{C}$,*

(b) *for each $\alpha, \beta \in \mathbb{N}^d$, there exist $c > 0$, $s_1 \geq 0$ and $s_2 \geq 0$ such that for all $q, x, y \in \mathbb{R}^d$:*

$$\left|\partial_x^\alpha \partial_y^\beta \gamma^B(q; x, y)\right| \leq c \langle x\rangle^{s_1} \langle y\rangle^{s_2}.$$

Proof. We use the explicit parameterized form of γ^B

$$\gamma^B(q;x,y) = \exp\Big\{ -i \sum_{j,k=1}^{d} x_j\, y_k \int_0^1 \Big[\int_0^1 s\, B_{jk}\big(q - \tfrac{1}{2}x - \tfrac{1}{2}y + sx + st(y-x)\big)\mathrm{d}s \Big] \mathrm{d}t \Big\}. \tag{7.4.10}$$

A careful examination of (7.4.10) leads directly to the results (a) and (b). See also the proof of Lemma 4.2 in [MPR05]. □

For the next statement, recall that $F_{a,j}(\cdot;\cdot,\cdot)$ has been introduced in (7.4.3).

Lemma 7.4.2. *For each $j \in \{1,\dots,d\}$, each $\alpha,\beta,\gamma \in \mathbb{N}^d$ and each $\mu > \max\{1,s\}$ there exists $c>0$ such that*

$$\big|\partial_p^\alpha\, \partial_k^\beta\, \partial_l^\gamma\, F_{a,j}(p;k,l)\big| \le c\, a^{-1/\mu}\, \langle p\rangle^{s/\mu - 1 - |\alpha|}\, \langle k\rangle^s\, \langle l\rangle^{2s} \tag{7.4.11}$$

for all $p,k,l \in \hat{\mathbb{R}}^d$ and $a \ge -\inf h + 1$.

Proof. It is enough to show that the expression

$$\sup_{t\in[0,1]} \Big|\partial_p^\alpha\, \partial_k^\beta\, \partial_l^\gamma \big[(l_j - k_j)\; (\partial_j h)\big(p + (t-1)l - tk\big)\; h_a^{-1}(p-l)\big]\Big| \tag{7.4.12}$$

is dominated by the r.h.s. term of (7.4.11) with a constant c not depending on p, k, l and a.

It is easy to see that for any $\delta \in \mathbb{N}^d$, we have $\partial^\delta h_a^{-1} = h_a^{-1}\, u_{a,\delta}$, where $u_{a,\delta} \in S^{-|\delta|}(\hat{\mathbb{R}}^d)$ uniformly in a. By using this, the Leibnitz formula and the inequality $\langle x+y\rangle \le \sqrt{2}\langle x\rangle\langle y\rangle$, it follows straightforwardly that (7.4.12) is dominated by

$$c_1 h_a^{-1}(p-l)\langle p\rangle^{s-1-|\alpha|}\langle k\rangle^s\langle l\rangle^s$$

for some $c_1 > 0$ independent of p, k, l and a. Furthermore, by using the ellipticity of h, we see that there exist $c_2 > 0$ and $c_3 > 0$ independent of p,l and a such that $h_a^{-1}(p-l) \le c_2\,\langle l\rangle^s[a + c_3\,\langle p\rangle^s]^{-1}$ for all $p,l \in \hat{\mathbb{R}}^d$. The final step consists in taking into account the inequality $a + c_3\langle p\rangle^s \ge \mu^{1/\mu}\,(\nu c_3)^{1/\nu}\, a^{1/\mu}\,\langle p\rangle^{s/\nu}$, valid for any $\mu \ge 1$, $\nu \ge 1$ with $\mu^{-1} + \nu^{-1} = 1$. □

In order to state the next lemma in its full generality, we need the definition:

Definition 7.4.3. *For $s \in \mathbb{R}$, $S^s(\hat{\mathbb{R}}^d;\mathscr{C})$ denotes the set of all functions $f : \mathbb{R}^d \times \hat{\mathbb{R}}^d \to \mathscr{C}$ that satisfy:*

(i) *$f(\cdot;p) \in \mathscr{C}$ for all $p \in \mathbb{R}^d$,*

(ii) *$f(q;\cdot) \in C^\infty(\hat{\mathbb{R}}^d)$, $\forall q \in \mathbb{R}^d$, and for each $\alpha \in \mathbb{N}^d$*

$$\sup_{q\in\mathbb{R}^d} \|f(q;\cdot)\|_{s,\alpha} := \sup_{q\in\mathbb{R}^d}\, \sup_{p\in\hat{\mathbb{R}}^d} \big[\langle p\rangle^{-s+|\alpha|}\,|\partial_p^\alpha f(q;p)|\big] < \infty\,.$$

It is easily seen that the algebraic tensor product $\mathscr{C} \odot S^s(\hat{\mathbb{R}}^d)$ is contained in $S^s(\hat{\mathbb{R}}^d;\mathscr{C})$.

Lemma 7.4.4. *Let f be an element of $S^t(\hat{\mathbb{R}}^d;\mathscr{C})$ with $t<0$. Then its partial Fourier transform $\mathfrak{F}^{-1}f$ is an element of $L^1(\mathbb{R}^d;\mathscr{C})$ that satisfies for a suitable large integer m*

$$\|\mathfrak{F}^{-1}f\|_{L^1(\mathbb{R}^d;\mathscr{C})} \le c \max_{|\alpha|\le m} \sup_{q\in\mathbb{R}^d} \|f(q;\cdot)\|_{t,\alpha} \,. \tag{7.4.13}$$

Proof. This is a straightforward adaptation of the proof of [ABG96, Prop. 1.3.3] (see also [ABG96, Prop. 1.3.6]). We decided to present it in order to put into evidence the explicit bound (7.4.13). Actually, the arguments needed to control the behavior in the variable q are easy and we leave them to the reader; we take simply $f\in S^t(\hat{\mathbb{R}}^d)$.

Since the case $t\le -d$ is rather simple, we shall concentrate on the more difficult one: $-d<t<0$. Let us first choose a cutoff function $\chi\in C_c^\infty(\mathbb{R}^d)$ that is 1 in a neighbourhood of 0. One has the estimates (with $\mathcal{F}$ the Fourier transform but without the constant factor):

$$\begin{aligned}
&\|(1-\chi)\mathcal{F}^{-1}f\|_{L^1} \;\le\; C\sum_{|\alpha|=m}\||Q|^{-2m}(1-\chi)\mathcal{F}^{-1}(\partial^{2\alpha}f)\|_{L^1}\\
\le\;& C\Big(\int_{\mathbb{R}^d}\big(1-\chi(x)\big)^2|x|^{-4m}\,\mathrm{d}x\Big)^{1/2}\sum_{|\alpha|=m}\|\partial^{2\alpha}f\|_{L^2}\\
\le\;& C'\Big(\int_{\mathbb{R}^d}\big(1-\chi(x)\big)^2|x|^{-4m}\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\hat{\mathbb{R}}^d}\langle p\rangle^{2(t-2m)}\,\mathrm{d}p\Big)^{1/2}\max_{|\alpha|=2m}\|f\|_{t,\alpha}\,,
\end{aligned}$$

where we take $m\in\mathbb{N}$ with $4m>d$ to make the integrals convergent.

We study now the behavior of $\mathcal{F}^{-1}f$ near the origin, a more difficult matter. Let us fix a second cutoff function $\varphi\in C^\infty(\hat{\mathbb{R}}^d)$ such that $0\le\varphi\le1$, $\varphi(p)=0$ for $|p|\le1$ and $\varphi(p)=1$ for $|p|\ge2$. For $b>0$ we set $\varphi_b(p):=\varphi(bp)$. We have:

$$\big|\big\{\mathcal{F}^{-1}\big((1-\varphi_b)f\big)\big\}(y)\big| \;\le\; \int_{|p|\le 2/b}|f(p)|\,\mathrm{d}p \;\le\; \|f\|_{t,0}\int_{|p|<2/b}|p|^t\,\mathrm{d}p \le C\,\|f\|_{t,0}\,b^{-d-t}.$$

Moreover, if $m\in2\mathbb{N}$ with $m\ge d+1$, then one has:

$$\begin{aligned}
&|y|^m|[\mathcal{F}^{-1}(\varphi_b f)](y)| \;\le\; C\sum_{|\alpha|=m}\big|\big[\mathcal{F}^{-1}\big(\partial^\alpha(\varphi_b f)\big)\big](y)\big|\\
\le\;& C\sum_{|\alpha|=m}\sum_{\beta\le\alpha}C_\alpha^\beta\, b^{|\alpha-\beta|}\int_{\hat{\mathbb{R}}^d}|(\partial^{\alpha-\beta}\varphi)(bp)|\,|(\partial^\beta f)(p)|\,\mathrm{d}p\\
\le\;& C'\max_{|\alpha|\le m}\|f\|_{t,\alpha}\Big\{\int_{|p|\ge1/b}|p|^{t-m}\,\mathrm{d}p+\sum_{|\beta|<m}b^{m-|\beta|}\int_{1/b<|p|<2/b}|p|^{t-|\beta|}\,\mathrm{d}p\Big\}\\
=\;& C''\max_{|\alpha|\le m}\|f\|_{t,\alpha}\,b^{m-d-t}.
\end{aligned}$$

By fixing $b := |y|$, we get $|[\mathcal{F}^{-1}(\varphi_{|y|} f)](y)| \leq C'' \max_{|\alpha| \leq m} \|f\|_{t,\alpha} \, |y|^{-d-t}$. The singularity at the origin is integrable, and putting all the inequalities together we obtain (7.4.13). $\square$

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