Chapter 7

Magnetic systems

In this chapter we shall see how all the previous constructions can be used when a magnetic field is considered on $\mathbb{R}^d$.

Very briefly, a *continuous magnetic field* is described by a closed continuous 2-form B defined on $\mathbb{R}^d$. It is well-known that any such field B may be written as the differential $\mathrm{d}A$ of a 1-forms A called a *vector potential*, which is highly non-unique (the gauge ambiguity). By using coordinates, one has

$$B_{jk} = \partial_j A_k - \partial_k A_j \qquad \text{for any } j,k \in \{1,\dots \mathrm{d}\}.$$

In the presence of the field $B = \mathrm{d}A$, the prescription (6.1.1) has to be modified. This topic was very rarely touched in the literature and the following wrong solution appears: The *minimal coupling principle* says roughly that the momentum D should be replaced with *the magnetic momentum* $\Pi^A = D - A(X)$. This originated in Lagrangian classical mechanics and works well also at the quantum level as long as we consider operators which are polynomials of order less or equal to 2. But if one just replaces in (6.1.1) the expression $f\big((x+y)/2,\eta\big)$ by $f\big((x+y)/2,\eta - A(x+y)/2\big)$ one gets a formula which misses the right gauge covariance. Indeed, let us denote the result of this procedure for some function f in phase space by $\mathfrak{Op}_A(f)$. If another vector potential A' is chosen such that $A' = A + \nabla\rho$ with ρ a scalar function, then $\mathrm{d}A' = \mathrm{d}A$. But the expected formula $\mathfrak{Op}_{A'}(f) = e^{i\rho}\mathfrak{Op}_A(f)e^{-i\rho}$ is verified for some simple cases (A, A' linear and f arbitrary, or f polynomial of order strictly less than 3 in η and A, A' arbitrary), but it fails in general.

Thus, the aim of the following sections is two show that the correct solution can directly be inferred from the formalism constructed before, without the invocation of a minimal coupling principle. The content of this chapter is borrowed from the three references [MPR05, MPR07, LMR10].

7.1 Magnetic twisted dynamical systems

From now on, the group G will always be $\mathbb{R}^d$, with its usual action θ by translations. The 2-cocycle will be defined in terms of the magnetic field. More precisely, the magnetic field on $\mathbb{R}^d$ is a closed continuous 2-form B. Since on $\mathbb{R}^d$ we have canonical global coordinates, we shall speak freely of the components B_{jk} of B; they are continuous real functions on $\mathbb{R}^d$ satisfying $B_{kj} = -B_{jk}$ and (in the distributional sense)

$$\partial_j B_{kl} + \partial_l B_{jk} + \partial_k B_{lj} = 0 \qquad \forall j, k, l \in \{1, \ldots, d\}.$$

It is well-known that $B = \mathrm{d}A$ for some 1-form A on $\mathbb{R}^d$, called a vector potential, which is highly non-unique. For simplicity, we shall consider only continuous A; this is always possible since at least one continuous vector potential always exists, namely *the transversal gauge* which is defined by

$$A_j(x) := -\sum_{k=1}^{d} \int_0^1 B_{jk}(sx) s x_k \,\mathrm{d}s. \tag{7.1.1}$$

Given a k-form C on $\mathbb{R}^d$ and a compact k-surface $\gamma \subset \mathbb{R}^d$, we define

$$\Gamma^C(\gamma) := \int_\gamma C,$$

this integral having a well-defined parametrization independent meaning. We shall mainly encounter circulations of 1-forms along linear segments $\gamma = [x, y]$ and fluxes of 2-forms through triangles $\gamma = \langle x, y, z\rangle$. In particular, for a continuous magnetic field B one defines

$$\omega^B(q; x, y) := e^{-i\Gamma^B(\langle q, q+x, q+x+y\rangle)} \qquad \text{for all } x, y, q \in \mathbb{R}^d. \tag{7.1.2}$$

From now on, let us fix a $\mathbb{R}^d$-algebra $\mathscr{C}$, *i.e.* a C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is invariant under the actions of $\mathbb{R}^d$ by translations. Note that in Definition 5.4.1 we have also assumed that $C_0(\mathbb{R}^d) \subset \mathscr{C}$, but that this additional condition is not necessary here. By Gelfand representation, we know that $\mathscr{C} \cong C_0(\Omega)$, with Ω the spectrum of $\mathscr{C}$. In this setting, the additional assumption $C_0(\mathbb{R}^d) \subset \mathscr{C}$ allowed one to identify $\mathbb{R}^d$ with a dense subset of Ω. Let us now consider $C(\Omega)$, the set of continuous functions on Ω. If $\mathscr{C}$ is not unital, then such functions can be unbounded. The simplest example is obtained by considering $\mathscr{C} = C_0(\mathbb{R}^d)$ with Ω equal to $\mathbb{R}^d$. Taking this observation into account, let us now define a magnetic field which is related to the $\mathbb{R}^d$-algebra $\mathscr{C}$:

Definition 7.1.1. *A magnetic field B is of type $\mathscr{C}$ with $\mathscr{C} \cong C_0(\Omega)$ if all its components $\{B_{jk}\}_{j,k=1}^d$ belong to $C(\Omega; \mathbb{R})$.*

Clearly, if $B_{jk} \in \mathscr{C}$ for any $j, k \in \{1, \ldots, d\}$, then B is a magnetic field of type $\mathscr{C}$. However, the previous definition is more general, and unbounded magnetic field can be considered in this setting. We recall that the notion of standard twisted system has been introduced in Definition 5.4.2.

Lemma 7.1.2. *If B is a magnetic field of type $\mathscr{C}$, then $(\mathscr{C}, \mathbb{R}^d, \theta, \omega^B)$ is a standard twisted dynamical system.*

Proof. The proof that ω^B is a normalized 2-cocyle, *i.e.* that it satisfies relations (5.1.1) and (5.1.2), follows easily by direct computations (for the first one use the Stokes Theorem for the closed 2-form B and the tetrahedron of vertices $q, q+x, q+x+y, q+x+y+z$).

We now show that ω^B has the right continuity properties. It should define a mapping

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \to \left[\omega^B(x,y)\right](\cdot) \equiv \omega^B(\cdot; x, y) \in C(\Omega; \mathbb{T}), \tag{7.1.3}$$

continuous with respect to the topology of uniform convergence on compact subsets of Ω. But this is equivalent to the fact that ω^B defines un element of $C(\Omega \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{T})$. Note that this type of statement already appeared in the proof of Lemma 5.3.3. Taking into account obvious properties of the exponential, this amounts to the fact that the function

$$\varphi^B : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad \varphi^B(q; x, y) := \Gamma^B(\langle q, q+x, q+x+y\rangle)$$

can be viewed as a continuous function on $\Omega \times \mathbb{R}^d \times \mathbb{R}^d$.

We use the parametrization

$$\varphi^B(q; x, y) = \sum_{j,k=1}^{d} x_j\, y_k \int_0^1 \int_0^1 s\, B_{jk}(q + sx + sty)\, \mathrm{d}s\, \mathrm{d}t.$$

Since the continuous action θ on $\mathscr{C}$ defines a continuous mapping θ on Ω, one has the continuous correspondence $\Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (q; x, y) \to q + sx + sty = \theta_{sx+sty}(q) \in \Omega$. Since B_{jk} is seen as a continuous function from Ω to $\mathbb{R}$, the assertion follows easily. □

Exercise 7.1.3. *Work out the details of the previous proof, and in particular show that ω^B satisfies the two conditions* (5.1.1) *and* (5.1.2).

From now on, we can call $(\mathscr{C}, \mathbb{R}^d, \theta, \omega^B)$ *the twisted dynamical system associated with the abelian algebra $\mathscr{C}$ and the magnetic field B*. In most of the cases the 2-cocycle $\omega^B \in Z^2\big(\mathbb{R}^d; \mathscr{U}(\mathscr{C})\big)$ is not trivial. But as Proposition 5.4.3 shows, it is pseudo-trivial. In fact, its pseudo-trivialization can be achieved by a vector potential. Any continuous 1-form A defines a 1-cochain $\lambda^A \in C^1\big(\mathbb{R}^d; C(\mathbb{R}^d; \mathbb{T})\big)$ via its circulation:

$$\left[\lambda^A(x)\right](q) \equiv \lambda^A(q; x) = e^{-i\Gamma^A([q, q+x])} = e^{-ix\cdot\int_0^1 A(q+sx)\,\mathrm{d}s}. \tag{7.1.4}$$

As soon as $\mathrm{d}A = B$, we have $\delta^1(\lambda^A) = \omega^B$ $\big($a priori with respect to $C(\mathbb{R}^d; \mathbb{T})\big)$, by a suitable version of Stokes Lemma. As said above, the transversal gauge offers a continuous vector potential corresponding to a given B. Actually, this is consistent with the choice (5.3.1) of a pseudo-trivialization of ω^B: for $q, x \in \mathbb{R}^d$, $\lambda(q; x) := \omega^B(0; q, x) = e^{-i\Gamma^B(\langle 0, q, q+x\rangle)}$ and it follows immediately that $\Gamma^B(\langle 0, q, q+x\rangle) = \Gamma^A([q, q+x])$, with A given by (7.1.1).

Since specific standard twisted dynamical systems can be constructed based on any magnetic field of type $\mathscr{C}$, the whole formalism of the preceding chapters is available. In particular, twisted crossed product algebra $\mathscr{C} \rtimes^{\omega^B}_{\theta,\tau} \mathbb{R}^d$, also denoted by $\mathscr{C} \rtimes^{B}_{\theta,\tau} \mathbb{R}^d$ and their Schrödinger representations are at hand. Note that as always, the dependence on τ is within isomorphism, and that for any continuous B the C^*-algebra $C_0(\mathbb{R}^d) \rtimes^{B}_{\theta,\tau} \mathbb{R}^d$ is isomorphic to $\mathscr{K}(\mathcal{H})$, the ideal of all compact operators in $\mathcal{H} = L^2(\mathbb{R}^d)$.

Let us close this section with some comments on the magnetic momentum, already introduced in the preamble of this chapter. The fact that the magnetic 2-cocycle ω^B satisfies

$$\omega^B(q; sx, tx) = 1, \quad \forall q, x \in \mathbb{R}^d \quad \text{and} \quad \forall s, t \in \mathbb{R} \tag{7.1.5}$$

leads directly to the magnetic momenta. Indeed, let us fix some continuous A such that $\mathrm{d}A = B$, and thus $\delta^1(\lambda^A) = \omega^B$. Then λ^A satisfies for all $q, x \in \mathbb{R}^d$ and all $s, t \in \mathbb{R}$: $\lambda^A(q; sx+tx) = \lambda^A(q; sx)\lambda^A(q+sx; tx)$ (note that in general, if λ is not the exponential of a circulation this will not be true). We consider then the Schrödinger covariant representation $(\mathcal{H}, \pi, U^A)$ with $\mathcal{H} = L^2(\mathbb{R}^d)$, $\pi(a) = a(X)$ and $U^A = U^{\lambda^A}$ defined by

$$[U^A_y u](x) \equiv [U^A(y)u](x) = \lambda^A(x; y)u(x+y), \quad x, y \in \mathbb{R}^d, \quad u \in \mathcal{H}.$$

The unitary operators $\{U^A(y)\}_{y\in\mathbb{R}^d}$ are called *the magnetic translations.* They often appear in the physical literature. One has, by a short computation,

$$U^A(sx + tx) = U^A(sx)U^A(tx), \quad \forall x \in \mathbb{R}^d, \quad \forall s, t \in \mathbb{R} \tag{7.1.6}$$

and this also implies $U^A(-x) = U^A(x)^{-1} = U^A(x)^*$ for all $x \in \mathbb{R}^d$. In fact, the formula

$$U^A(y)U^A(z) = \pi[\omega^B(y, z)]U^A(y + z), \quad y, z \in \mathbb{R}^d$$

shows that (7.1.6) is equivalent with (7.1.5). For $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, let us set $U^A_t(x) := U^A(tx)$. By (7.1.6), we observe that $\{U^A_t(x)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group in $\mathcal{H}$ for any x. Thus, by Stone Theorem (see Theorem 1.7.12), it has a self-adjoint generator that moreover depends linearly (as a linear operator on $\mathcal{H}$) on the vector $x \in \mathbb{R}^d$. Thus we denote it by $x\cdot\Pi^A$ and call it *the projection on x of the magnetic momentum associated with the vector potential A*. For any index $j \in \{1, ..., n\}$ we set $\Pi^A_j := e_j \cdot \Pi^A$ the projection of the magnetic momentum on the j'th vector of the canonical base in $\mathbb{R}^d$. A direct computation shows that on $C^\infty_c(\mathbb{R}^d)$ one has $\Pi^A_j = D_j - A_j(X)$.

7.2 Magnetic pseudodifferential calculus

In this section, we adapt the results presented in Section 6.1 when a magnetic field is also present. Most of the following formulas appeared already in the more general setting of Section 6.2, but this section can be seen as a useful résumé for the interested reader.

Let us directly start by introducing the analog of the Weyl system recalled in (6.1.4) but in the presence of a magnetic field. For the time being, B is any continuous magnetic field on $\mathbb{R}^d$ and A is any corresponding continuous vector potential. Associated with the Schrödinger covariant representation $(\mathcal{H}, \pi, U^A)$ defined above, we can now define *the magnetic Weyl system* W^A by

$$\Xi \ni \mathsf{x} \mapsto W^A\big(\mathsf{x}\big) := e^{-\frac{i}{2}x\cdot\xi} V_\xi U^A(x) \in \mathscr{U}(\mathcal{H}).$$

These unitary operators satisfy then the relations

$$W^A(\mathsf{x})\, W^A(\mathsf{y}) = e^{\frac{i}{2}\sigma(\mathsf{x},\mathsf{y})}\, \pi[\omega^B(x,y)]\, W^A(\mathsf{x}+\mathsf{y})$$

for any $\mathsf{x} = (x,\xi)$ and $\mathsf{y} = (y,\eta)$.

Exercise 7.2.1. *Check the above relations*

For any $f \in \mathcal{S}(\Xi)$ we can then write explicitly the operator $\mathfrak{Op}^A(f) := \mathfrak{Op}^{\lambda^A}_{1/2}(f)$ in $\mathcal{H}$ which has been introduced in Proposition 6.2.2, namely

$$\big[\mathfrak{Op}^A(f)u\big]\,(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\hat{\mathbb{R}}^d} e^{i(x-y)\cdot\eta} e^{-i\Gamma^A([x,y])} f\left(\frac{x+y}{2},\eta\right) u(y)\,\mathrm{d}y\,\mathrm{d}\eta.$$

Note that this formula can be called *the magnetic Weyl calculus*. Furthermore, it is easily observed that this is an integral operator with kernel

$$K^A := \tilde{\lambda}^A\, S^{-1}\,(\mathbf{1}\otimes\overline{\mathcal{F}}_{\hat{\mathbb{R}}^d})$$

where $\tilde{\lambda}^A(x,y) := \lambda^A(x;y-x)$ and $\big(S^{-1}h\big)(x,y) = h\left(\frac{x+y}{2}, x-y\right)$. With this formula, we can now extend the map K^A and thus define $\mathfrak{Op}^A(F)$ for any $F \in \mathcal{S}'(\Xi)$ as the integral operator with kernel $K^A(F)$, defined on $\mathcal{S}(\mathbb{R}^d)$ with values in $\mathcal{S}'(\mathbb{R}^d)$. It seems legitimate to view the correspondence $f \to \mathfrak{Op}^A(f)$ as a functional calculus for the family of self-adjoint operators $X_1,\dots,X_d,\Pi^A_1,\dots,\Pi^A_d$. The high degree of non-commutativity of these $2d$ operators stays at the origin of the sophistication of the symbolic calculus. The commutation relations

$$i[X_j,X_k] = 0,\quad i[\Pi^A_j,X_k] = \delta_{jk},\quad i[\Pi^A_j,\Pi^A_k] = -B_{jk}(X),\qquad j,k = 1,\dots,d \tag{7.2.1}$$

collapse for $B = 0$ to the canonical commutation relations satisfied by X and D, see Exercise 4.1.3. But they are much more complicated, especially when B is not a polynomial. The main mathematical miracle that allows, however, a nice treatment is the fact that (7.2.1) can be recast in the form of a covariant representation of a twisted dynamical system.

Let us stress once more that the functional calculus that we have defined is gauge covariant, in the sense that it satisfies the property: If $A' = A + \nabla\varphi$ with $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous, then $\mathfrak{Op}^{A'}(f) = e^{i\varphi(X)}\mathfrak{Op}^A(f)e^{-i\varphi(X)}$. This gauge covariance property may be seen as a special instance of Proposition 6.2.2.

The extension of the usual Moyal product has a particular form in the magnetic setting. More precisely, by adapting the formula obtained in Section 6.2 to the magnetic 2-cocyle and for $\tau = 1/2$, one obtains on $\mathcal{S}(\Xi)$ the composition and the involution:

$$(f \circ^B g)(\mathsf{x}) = \frac{4^d}{(2\pi)^d}\int_\Xi\int_\Xi e^{-2i\sigma(\mathsf{x}-\mathsf{y},\mathsf{x}-\mathsf{z})}e^{-i\Gamma^B(\langle x-z+y,y-x+z,z-y+x\rangle)}f(\mathsf{y})g(\mathsf{z})\,\mathrm{d}\mathsf{y}\,\mathrm{d}\mathsf{z}, \quad (7.2.2)$$

with $\mathsf{x},\mathsf{y},\mathsf{z} \in \Xi$, and

$$f^{\circ^B}(\mathsf{x}) = \overline{f(\mathsf{x})}, \quad \forall \mathsf{x} \in \Xi.$$

Note that with these formulas, one has $(f \circ^B g)^{\circ^B} = g^{\circ^B} \circ^B f^{\circ^B}$ as well as

$$\mathfrak{Op}^A(f \circ^B g) = \mathfrak{Op}^A(f)\mathfrak{Op}^A(g), \qquad \text{and} \qquad \mathfrak{Op}^A(f^{\circ^B}) = \mathfrak{Op}^A(f)^*.$$

Exercise 7.2.2. *Without relying on the content of the previous sections, check directly these equalities.*

We remark that the involution $^{\circ^B}$ and the product $\circ^B$ are defined intrinsically, without any choice of a vector potential. The choice is only needed when we represent the resulting structures on the Hilbert space $L^2(\mathbb{R}^d)$. We call (7.2.2) *the magnetic Moyal product.* The involution $^{\circ^B}$ does not depend on B at all. This is no longer true if $\tau \neq 1/2$. The property $\omega^B(x,-x) = 1$, $\forall x \in \mathbb{R}^d$, is also used to get the simple form of $^{\circ^B}$.

Let us now assume that B is of type $\mathscr{C}$ for some $\mathbb{R}^d$-algebra $\mathscr{C}$. The C^*-algebra $\mathfrak{C}^{\omega^B}_{\mathscr{C},1/2}$, introduced in Section 6.2, will be denoted by $\mathfrak{C}^B_{\mathscr{C}}$. We call it *the C^*-algebra of pseudodifferential symbols of class $\mathscr{C}$ associated with B.* We recall that it is essentially a partial Fourier transform of the twisted crossed product $\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d$. The formulas defining the magnetic Weyl calculus make sense at least on the dense subset $(\mathbf{1} \otimes \overline{\mathcal{F}}_{\mathbb{R}^d})L^1(\mathbb{R}^d;\mathscr{C})$, with iterated integrals. The extension of $\mathfrak{Op}^A$ is a faithful representation of the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$ for any continuous A with $\mathrm{d}A = B$. If $C_0(\mathbb{R}^d) \subset \mathscr{C}$, then $\mathfrak{Op}^A$ is irreducible.

We close this section with some arguments about one possible extension for the product $\circ^B$. Indeed, as already mentioned in the Extension 6.2.4, the integrals defining $f \circ^B g$ are absolutely convergent only for restricted classes of symbols. In order to deal with more general distributions, an extension by duality was proposed in [MP04] under an additional smoothness condition on the magnetic field. So let us assume that the components of the magnetic field are $C^\infty_{pol}(\mathbb{R}^d)$-functions, *i.e.* they are indefinitely derivable and each derivative is polynomially bounded. The duality approach is based on the observation [MP04, Lem. 14] : For any f, g in the Schwartz space $\mathcal{S}(\Xi)$, we have

$f \circ^B g \in \mathcal{S}(\Xi)$, and

$$\int_\Xi [f \circ^B g](\mathsf{x}) \mathrm{d}\mathsf{x} = \int_\Xi [g \circ^B f](\mathsf{x}) \mathrm{d}\mathsf{x} = \int_\Xi f(\mathsf{x})\, g(\mathsf{x}) \mathrm{d}\mathsf{x} = \langle f, \overline{g} \rangle =: (f, g).$$

As a consequence, by using the associativity of $\circ^B$ and the symmetry of $(\cdot,\cdot)$, one easily deduces that for $f, g, h \in \mathcal{S}(\Xi)$, one has

$$(f \circ^B g, h) = (f, g \circ^B h) = (g, h \circ^B f).$$

Definition 7.2.3. *For any distribution $F \in \mathcal{S}'(\Xi)$ and any function $f, h \in \mathcal{S}(\Xi)$ we define*

$$(F \circ^B f, h) := (F, f \circ^B h), \qquad (f \circ^B F, h) := (F, h \circ^B f)$$

The expressions $F \circ^B f$ and $f \circ^B F$ are *a priori* tempered distributions. The Moyal algebra is precisely the set of elements of $\mathcal{S}'(\Xi)$ that preserves regularity by composition.

Definition 7.2.4. The magnetic Moyal algebra $\mathscr{M}(\Xi)$ *is defined by*

$$\mathscr{M}(\Xi) := \left\{ F \in \mathcal{S}'(\Xi) \mid F \circ^B f \in \mathcal{S}(\Xi) \text{ and } f \circ^B F \in \mathcal{S}(\Xi) \text{ for all } f \in \mathcal{S}(\Xi) \right\}.$$

For two distributions F and G in $\mathscr{M}(\Xi)$, the magnetic Moyal product *can be extended by*

$$(F \circ^B G, h) := (F, G \circ^B h) \quad \text{for all } h \in \mathcal{S}(\Xi).$$

Clearly, the set $\mathscr{M}(\Xi)$ with this composition law and the complex conjugation $F \mapsto F^\circ$ is a unital *-algebra. An important result [MP04, Prop. 23] concerning the Moyal algebra is that it contains $C^\infty_{pol,u}(\Xi)$, the space of infinitely derivable complex functions on Ξ having uniform polynomial growth at infinity. Finally let us quote a result linking $\mathscr{M}(\Xi)$ with the functional calculus $\mathfrak{Op}^A$ [MP04, Prop. 21] : For any vector potential A belonging to $C^\infty_{pol}(\mathbb{R}^d)$, $\mathfrak{Op}^A$ is an isomorphism of *-algebras between $\mathscr{M}(\Xi)$ and $\mathscr{B}[\mathcal{S}(\mathbb{R}^d)] \cap \mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$, where $\mathscr{B}[\mathcal{S}(\mathbb{R}^d)]$ and $\mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$ are, respectively, the spaces of linear continuous operators on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

Remark 7.2.5. *The extension by duality also gives compositions $\mathscr{M}(\Xi) \circ^B \mathcal{S}'(\Xi) \subset \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \circ^B \mathscr{M}(\Xi) \subset \mathcal{S}'(\Xi)$. One checks plainly that associativity holds for any three factors product with two factors belonging to $\mathscr{M}(\Xi)$ and one in $\mathcal{S}'(\Xi)$.*

7.3 Magnetic Schrödinger operators

From now on, we consider for simplicity a $\mathbb{R}^d$-algebra $\mathscr{C}$ which is unital and which contains $C_0(\mathbb{R}^d)$. As a consequence, $\mathscr{C} \cong C(\Omega)$ with Ω a compactification of $\mathbb{R}^d$. Then, given a magnetic field B of type $\mathscr{C}$, *cf.* Definition 7.1.1, a continuous vector potential A that generates B and a suitable symbol $h : \hat{\mathbb{R}}^d \to \mathbb{R}$, our aim is to show that the magnetic Schrödinger operator $h(\Pi^A)$ (which needs to be carefully defined) defines an observable affiliated to the C^*-algebra

$$\mathfrak{Op}^A(\mathfrak{C}^B_{\mathscr{C}}) = \mathfrak{Rep}^A\big(\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d\big) \equiv \mathfrak{Rep}^{\lambda^A}_{1/2}\big(\mathscr{C} \rtimes^B_{\theta,1/2} \mathbb{R}^d\big) \subset \mathscr{B}(\mathcal{H}),$$

see Definition 4.3.7 for the precision notion of affiliation. The proof of such a statement is rather difficult and we shall do it under some smoothness conditions on the magnetic field B and on the symbol h. We point out that we prove in fact in Theorem 7.3.2 a stronger result that does not depend on the choice of any particular vector potential.

Definition 7.3.1. *(i) For $s \in \mathbb{R}$, a function $h \in C^\infty(\hat{\mathbb{R}}^d)$ is* a symbol of type s, *written $h \in S^s(\hat{\mathbb{R}}^d)$, if the following condition is satisfied:*

$$\forall \alpha \in \mathbb{N}^d,\ \exists c_\alpha > 0 \ \text{ such that } \ |(\partial^\alpha h)(\xi)| \le c_\alpha \langle \xi \rangle^{s-|\alpha|} \text{ for all } \ \xi \in \hat{\mathbb{R}}^d.$$

(ii) The symbol h is called elliptic *if there exist $R > 0$ and $c > 0$ such that*

$$c\,\langle \xi \rangle^s \le h(\xi) \ \text{ for all } \xi \in \hat{\mathbb{R}}^d \ \text{ and } \ |\xi| \ge R.$$

We denote by $S^s_{el}(\hat{\mathbb{R}}^d)$ the family of elliptic symbols of type s, and set $S^\infty_{el}(\hat{\mathbb{R}}^d) := \cup_s S^s_{el}(\hat{\mathbb{R}}^d)$. Note that all the classes $S^s(\hat{\mathbb{R}}^d)$ are naturally contained in $C^\infty_{pol,u}(\Xi)$, thus in $\mathscr{M}(\Xi)$. For any $z \in \mathbb{C} \setminus \mathbb{R}$, we also set $r_z : \mathbb{R} \to \mathbb{C}$ by $r_z(\cdot) := (\cdot - z)^{-1}$.

We are in a position to state the main results about affiliation. The proofs of these statements are postponed until the next section.

Theorem 7.3.2. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Then each real $h \in S^\infty_{el}(\hat{\mathbb{R}}^d)$ defines an observable Φ^B_h affiliated to $\mathfrak{C}^B_{\mathscr{C}}$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$ one has*

$$(h - z) \circ^B \Phi^B_h(r_z) = 1 = \Phi^B_h(r_z) \circ^B (h - z). \tag{7.3.1}$$

In fact one even has $\Phi^B_h(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathcal{S}'(\Xi)$, so the compositions can be interpreted as $\mathscr{M}(\Xi) \times \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \times \mathscr{M}(\Xi) \to \mathcal{S}'(\Xi)$.

We shall now consider a scalar potential $V \in \mathscr{C}$. As seen in Theorem 3.4.5 the algebra $\mathscr{C}$ can be identified with part of the multiplier algebra of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Then, a straightforward reformulation of the perturbative argument presented in [ABG96, p. 365–366] allows one to define the observable $\Phi^B_{h,V} := \Phi^B_h + V$. Considering now $h + V \in \mathcal{S}'(\Xi)$ we remark that we can compute the Moyal product

$$(h + V - z) \circ^B \Phi^B_{h,V}(r_z) = (h - z) \circ^B \Phi^B_{h,V}(r_z) + V \circ^B \Phi^B_{h,V}(r_z) = 1$$

by using the explicit formula of $\Phi^B_{h,V}$ given in [ABG96, p. 366]. This leads then to the following statement:

Corollary 7.3.3. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Let also V be a real function in $\mathscr{C}$. Then $\Phi^B_{h,V}$ is an observable affiliated to $\mathfrak{C}^B_{\mathscr{C}}$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$ one has*

$$(h + V - z) \circ^B \Phi^B_{h,V}(r_z) = 1 = \Phi^B_{h,V}(r_z) \circ^B (h + V - z).$$

These statements are elegant, being abstract, but in applications one also needs the represented version:

Corollary 7.3.4. *Assume that B is a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$, and let V be a real function in $\mathscr{C}$. Let A be a continuous vector potential that generates B. Then $\mathfrak{Op}^A(h) + V(X)$ defines a self-adjoint operator in $\mathcal{H}$ with domain given by the image of the operator $\mathfrak{Op}^A\left[(h-z)^{-1}\right]$ (which do not depend on $z \in \mathbb{C} \setminus \mathbb{R}$). This operator is affiliated to $\mathfrak{Op}^A(\mathfrak{C}^B_{\mathscr{C}})$.*

We finally give a description of the essential spectrum of the observables affiliated to the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$. For the generalized magnetic Schrödinger operators of Theorem 7.3.2, this is expressed in terms of the spectra of so-called *asymptotic operators*. The affiliation criterion and the algebraic formalism introduced above play an essential role in the proof of this result. Note that we shall mimic the approach already used in Section 4.5 in the absence of a magnetic field, and freely use the notations and concepts introduced there.

Recall that $\mathscr{C} \cong C(\Omega)$ with Ω a compactification of $\mathbb{R}^d$. Then, for any $\tau \in \Omega \setminus \mathbb{R}^d$, one sets $\mathcal{O}_\tau$ for the orbit generated by τ, and $\mathcal{Q}_\tau$ for the corresponding quasi-orbit. In this setting, for any $f \in C(\Omega)$, the function $x \mapsto f\big(\theta_x(\tau)\big)$ is an element of $BC_u(\mathbb{R}^d)$, see Exercise 4.5.2 for details. In particular, this construction holds for V and B_{jk} if both belong to $\mathscr{C}$.

Theorem 7.3.5. *Let B be a magnetic field whose components belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$ and let $V \in \mathscr{C}$ be a real function. Assume that $\{\mathcal{Q}_{\tau_i}\}_i$ is a covering of $\partial\Omega$ by quasi-orbits. Then for each real $h \in S^\infty_{el}(\hat{\mathbb{R}}^d)$ one has*

$$\sigma_{ess}\big[\mathfrak{Op}^A(h) + V(X)\big] = \overline{\cup_i \sigma\big[\mathfrak{Op}^{A_i}(h) + V_i(X)\big]}, \tag{7.3.2}$$

where A, A_i are continuous vector potentials for B, $B_i \equiv B|_{\mathcal{Q}_{\tau_i}}$, and $V_i \equiv V|_{\mathcal{Q}_{\tau_i}}$.

Clearly, the computation of the essential spectrum is first performed at an abstract level, *i.e.* without using any representation. This computation is more simple since no vector potentials are involved. Only for convenience and tradition, the previous represented version is also stated. Note also that the proof of this theorem is similar to the one presented in Section 4.5, the 2-cocycles fitting very well with the functoriality of the crossed products. We do not give any details here and refer to [MPR07, Sec. 3] for the interested reader.

7.4 Affiliation in the magnetic case

In this section we provide the proofs of Theorem 7.3.2 and of Corollary 7.3.4. Some technical arguments are postponed to the end of the section. Throughout the section, we assume tacitly all the assumptions of Theorem 7.3.2.

The proof of Theorem 7.3.2 will be based on the following strategy: Let $\mathscr{M}$ be an associative algebra with a composition law denoted by $\circ$ and let $\mathfrak{h}$ be an element of $\mathscr{M}$. Our aim is to find the inverse for $\mathfrak{h}$. Assume that $\mathfrak{h}'$ is another element such that $\mathfrak{h}\circ\mathfrak{h}'$ and $\mathfrak{h}'\circ\mathfrak{h}$ are invertible. These inverses are written $(\mathfrak{h}\circ\mathfrak{h}')^{(-1)}$ and $(\mathfrak{h}'\circ\mathfrak{h})^{(-1)}$ respectively. Then, the element $\mathfrak{h}'\circ(\mathfrak{h}\circ\mathfrak{h}')^{(-1)}$ is obviously a right inverse for $\mathfrak{h}$ and the element $(\mathfrak{h}'\circ\mathfrak{h})^{(-1)}\circ\mathfrak{h}'$ a left inverse for $\mathfrak{h}$. Both expressions are thus equal to $\mathfrak{h}^{(-1)}$.

In the sequel, we shall take for $\mathfrak{h}$ the strictly positive symbol $h+a$, with a large enough, and for $\mathfrak{h}'$ its pointwise inverse $(h+a)^{-1}$. Finding an inverse $(h+a)^{(-1)}$ for $h+a$ with respect to the composition law $\circ^B$ will lead rather easily to an observable. In the calculations below we shall use tacitly the some approximation procedures. For several arguments we will be forced to get out of the algebra $\mathscr{M}=\mathscr{M}(\Xi)$. This will be easily dealt with by a suitable use of elements of $\mathcal{S}'(\Xi)$.

Note finally that for simplicity, elements of $\hat{\mathbb{R}}^d$ will be denoted by p,k or l.

Proof of Theorem 7.3.2. (i) Let us consider an elliptic symbol h of order s and fix some real number $a\geq -\inf h+1$. We set $h_a:=h+a$, and denote by h_a^{-1} its inverse with respect to pointwise multiplication, *i.e.* $h_a^{-1}(p):=(h(p)+a)^{-1}$ for all $p\in\hat{\mathbb{R}}^d$. It is clear that h_a^{-1} is a symbol of type $-s$. Since both functions h_a and h_a^{-1} belong to $C^\infty_{pol,u}(\Xi)$, and thus to the Moyal algebra $\mathscr{M}(\Xi)$, one can calculate their product. By using (7.2.2) we obtain

$$\big(h_a\circ^B h_a^{-1}\big)(q,p)=\frac{4^d}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}x\int_{\hat{\mathbb{R}}^d}\mathrm{d}k\int_{\mathbb{R}^d}\mathrm{d}y\int_{\hat{\mathbb{R}}^d}\mathrm{d}l\,e^{-2i(k\cdot y-l\cdot x)}\gamma^B(q;2x,2y)\frac{h_a(p-k)}{h_a(p-l)},\tag{7.4.1}$$

with

$$\gamma^B(q;2x,2y):=\omega^B\big(q-x-y;2x,2(y-x)\big).\tag{7.4.2}$$

The last factor in the integral does not depend on x and y; it can be developed:

$$\frac{h_a(p-k)}{h_a(p-l)}=1+\sum_{j=1}^d(l_j-k_j)\frac{\int_0^1\mathrm{d}t\,(\partial_j h)\big(p-l+t(l-k)\big)}{h(p-l)+a}=:1+\sum_{j=1}^d F_{a,j}(p;k,l)\,.\tag{7.4.3}$$

Moreover, let

$$\tilde{\gamma}^B(q;k,l)\equiv(\mathbb{F}\gamma^B)(q;k,l):=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\,e^{-ik\cdot y}\,e^{il\cdot x}\,\gamma^B(q;x,y).$$

Then the following equality holds (in the sense of distributions):

$$\int_{\hat{\mathbb{R}}^d}\mathrm{d}k\int_{\hat{\mathbb{R}}^d}\mathrm{d}l\,\tilde{\gamma}^B(q;k,l)=\gamma^B(q;0,0)=1.\tag{7.4.4}$$

Thus, by inserting (7.4.3) and (7.4.4) into (7.4.1), we obtain

$$h_a \circ^B h_a^{-1} = 1 + \sum_{j=1}^{d} f_{a,j},$$

with

$$f_{a,j}(q;p) := \int_{\hat{\mathbb{R}}^d} \mathrm{d}k \int_{\hat{\mathbb{R}}^d} \mathrm{d}l\ \tilde{\gamma}^B(q;k,l)\, F_{a,j}(p;k,l) = \big\langle (\mathbb{F}\gamma^B)(q;\cdot,\cdot), F_{a,j}(p;\cdot,\cdot)\big\rangle. \quad (7.4.5)$$

The last notation is used in order to emphasize the duality between $C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$ and its dual. Indeed, for q,p fixed, Lemma 7.4.2 proves that $F_{a,j}(p;\cdot,\cdot) \in C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)$, and Lemma 7.4.1 proves that $\gamma^B(q;\cdot,\cdot) \in C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d)$, from which one infers that $(\mathbb{F}\gamma^B)(q;\cdot,\cdot) \in \big[C^\infty_{pol,u}(\hat{\mathbb{R}}^d \times \hat{\mathbb{R}}^d)\big]'$, see [Sch73, Chap. VII, Thm. XV].

(ii) We are now going to deduce some useful estimates on $f_{a,j}$. We set $\langle D_x\rangle \equiv \langle -i\partial_x\rangle$. For α, j fixed and m,n integers that we shall choose below, one has

$$\begin{aligned}|(\partial_p^\alpha f_{a,j})(q;p)| \le \sup_{x,y\in\mathbb{R}^d} |\langle x\rangle^{-n}\langle y\rangle^{-n}\langle D_x\rangle^m\langle D_y\rangle^m \gamma^B(q;x,y)| \cdot \\ \big\|\langle x\rangle^{-d}\langle y\rangle^{-d}\big\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \big\|\langle D_k\rangle^{n+d}\langle D_l\rangle^{n+d}\langle k\rangle^{-m}\langle l\rangle^{-m} \left(\partial_p^\alpha F_{a,j}\right)(p;\cdot,\cdot)\big\|_{L^2(\hat{\mathbb{R}}^d\times\hat{\mathbb{R}}^d)} \cdot\end{aligned} \quad (7.4.6)$$

By taking into account (7.4.11), and by some simple computations, one can fix m such that the last factor of (7.4.6) is dominated by $c_n\, a^{-1/\mu}\,\langle p\rangle^{s/\mu-1-|\alpha|}$, with $\mu > \max\{1,s\}$. Then, by using Lemma 7.4.1, one can choose n (depending on m) such that the first factor on the r.h.s. term of (7.4.6) is bounded. Altogether, one obtains

$$|(\partial_p^\alpha f_{a,j})(q;p)| \le c\, a^{-1/\mu}\,\langle p\rangle^{s/\mu-1-|\alpha|}, \quad (7.4.7)$$

where c depends on α and j but not on p,q or a.

(iii) Let us now show that for each j, $\mathfrak{F}^{-1}(f_{a,j})$ is an element of $L^1(\mathbb{R}^d;\mathscr{C})$, and thus belongs to the C^*-algebra $\mathfrak{C}^B_{\mathscr{C}}$.

By taking into account Lemma 7.4.1, the r.h.s. of the equation (7.4.5) can be rewritten as $\big\langle \gamma^B(q;\cdot,\cdot), (\mathbb{F}^*F_{a,j})(p,\cdot,\cdot)\big\rangle$, in which the duality between $C^\infty_{pol}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\big(C^\infty_{pol}(\mathbb{R}^d\times\mathbb{R}^d)\big)' = \mathbb{F}^*C^\infty_{pol,u}(\hat{\mathbb{R}}^d\times\hat{\mathbb{R}}^d)$ is emphasized. As γ^B defines a function from $\mathbb{R}^d\times\mathbb{R}^d$ to $\mathscr{C}$ (see Lemma 7.4.1) that is of class $C^\infty_{pol}(\mathbb{R}^d\times\mathbb{R}^d)$, we can easily prove that $f_{a,j}(\cdot;p)$ belongs to $\mathscr{C}$, for all $p \in \hat{\mathbb{R}}^d$ (by using partitions of unity on $\mathbb{R}^d\times\mathbb{R}^d$ and by approximating the duality pairing with finite linear combinations of elements in $\mathscr{C}$).

This observation together with (7.4.7) imply that the hypotheses of Lemma 7.4.4 are fulfilled for each $f_{a,j}$, with $t = -(1-s/\mu) < 0$. It follows that $\mathfrak{F}^{-1}(f_{a,j})$ belongs to $L^1(\mathbb{R}^d;\mathscr{C})$ and that there exists $C>0$ such that

$$\|\mathfrak{F}^{-1}(f_{a,j})\|_1 \le C\, a^{-1/\mu}.$$

Thus, for a large enough, the strict inequality $\big\| \sum_{j=1}^d \mathfrak{F}^{-1}(f_{a,j})\big\|_1 < 1$ holds. It follows that $\mathfrak{F}^{-1}(1+\sum_{j=1}^d f_{a,j})$ is invertible in $\widetilde{L^1}$, the minimal unitization of $L^1(\mathbb{R}^d;\mathscr{C})$. Equivalently, $h_a \circ^B h_a^{-1} \equiv 1 + \sum_{j=1}^d f_{a,j}$ is invertible in $\widetilde{\mathfrak{F}(L^1)}$, the minimal unitization of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Its inverse will be denoted by $\big(h_a \circ^B h_a^{-1}\big)^{(-1)}$.

(iv) We recall that $h_a^{-1} \in S^{-s}(\hat{\mathbb{R}}^d)$. Then, by Lemma 7.4.4 we get that $h_a^{-1} \in \mathfrak{F}\big(L^1(\mathbb{R}^d)\big) \subset \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Thus $h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)}$ is a well defined element of $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$. Moreover, one readily gets $h_a \circ^B [h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)}] = 1$. For this, just think of h_a and h_a^{-1} as elements of the Moyal algebra $\mathscr{M}(\Xi)$ and interpret $(h_a \circ^B h_a^{-1})^{(-1)} \in \widetilde{\mathfrak{F}(L^1)}$ as an element of $\mathcal{S}'(\Xi)$. The needed associativity follows easily from the definition by duality of the composition law as stated in Remark 7.2.5. In the same way one obtains $[(h_a^{-1} \circ^B h_a)^{(-1)} \circ^B h_a^{-1}] \circ^B h_a = 1$ in $\mathscr{M}(\Xi)$. In conclusion, there exists $a_0 \geq -\inf h + 1$ such that for any $a > a_0$ the symbol h_a possess an inverse with respect to the Moyal product

$$h_a^{(-1)} := h_a^{-1} \circ^B (h_a \circ^B h_a^{-1})^{(-1)} = (h_a^{-1} \circ^B h_a)^{(-1)} \circ^B h_a^{-1} \in \mathcal{S}'(\Xi)$$

that also belongs to $\mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathfrak{C}^B_{\mathscr{C}}$. The second equality follows from Remark 7.2.5 by straightforward arguments.

(v) We define $\Phi_h^B(r_x) := h_{-x}^{(-1)}$ for $x < -a_0$. Then $\Phi_h^B(r_x) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big) \subset \mathfrak{C}^B_{\mathscr{C}} \cap \mathcal{S}'(\Xi)$, its norm is uniformly bounded for x in the given domain and $(h-x) \circ^B \Phi_h^B(r_x) = \Phi_h^B(r_x) \circ^B (h-x) = 1$, as shown above. This allows us to obtain an extension to the half-strip $\{z = x+iy \mid x < -a_0, |y| < \delta\}$ for some $\delta > 0$ by setting

$$\Phi_h^B(r_z) := \Phi_h^B(r_x) \circ^B \{1 + (x-z)\Phi_h^B(r_x)\}^{(-1)}. \tag{7.4.8}$$

It follows that

$$(h-z) \circ^B \Phi_h^B(r_z) = \{(h-x) \circ^B \Phi_h^B(r_x) + (x-z)\Phi_h^B(r_x)\} \circ^B \{1 + (x-z)\Phi_h^B(r_x)\}^{(-1)} = 1.$$

We now prove that the map

$$\{z = x+iy \mid x < -a_0, |y| < \delta\} \ni z \mapsto \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d;\mathscr{C})\big)$$

satisfies the resolvent equation. Let us choose two complex numbers z and z' in this domain and subtract the two equations

$$(h-z) \circ^B \Phi_h^B(r_z) = 1, \qquad (h-z') \circ^B \Phi_h^B(r_{z'}) = 1 \tag{7.4.9}$$

in order to get $(h-z) \circ^B \{\Phi_h^B(r_z) - \Phi_h^B(r_{z'})\} + (z'-z)\Phi_h^B(r_{z'}) = 0$. By multiplying at the left with $\Phi_h^B(r_z)$ and by using the associativity, we obtain the resolvent equation

$$\Phi_h^B(r_z) - \Phi_h^B(r_{z'}) = (z-z')\Phi_h^B(r_z) \circ^B \Phi_h^B(r_{z'}).$$

Now, setting $z' = \overline{z} = x - iy$ with $y > 0$ and taking norms we get

$$\|\Phi_h^B(r_z)\|_{\mathfrak{F}(L^1(\mathbb{R}^d;\mathscr{C}))} \leq y^{-1}.$$

With this estimate and formula (7.4.8), the function $z \mapsto \Phi_h^B(r_z)$ can be extended to the domain $\mathbb{C} \setminus [-a_0, +\infty)$, preserving the relations (7.4.9). The resolvent equation may be proved in a similar way to hold on the entire domain $\mathbb{C} \setminus [-a_0, +\infty)$ and analyticity of the defined function follows in an evident way.

(vi) Thus we have got an analytic map $\mathbb{C}\setminus[-a_0, +\infty) \ni z \to \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(\mathbb{R}^d; \mathscr{C})\big)$ satisfying the resolvent equation and the symmetry condition. A general argument presented in [ABG96, p. 364] allows now to extend in a unique way the map Φ_h^B to a C^*-algebra morphism $C_0(\mathbb{R}) \to \mathfrak{C}_{\mathscr{C}}^B$. □

We can now provide the represented version of our affiliation criterion.

Proof of Corollary 7.3.4. We shall first consider the case $V = 0$ and then add V as a bounded perturbation.

Let us denote by D_z the range of the operator $\mathfrak{Op}^A[\Phi_h^B(r_z)] \in \mathscr{B}(\mathcal{H})$. By the resolvent identity it follows immediately that it is a subspace of $\mathcal{H}$ that does not depend on $z \in \mathbb{C} \setminus \mathbb{R}$. Thus we set $\mathsf{D}_z \equiv \mathsf{D}$. Since $h \in \mathscr{M}(\Xi)$, one has $\mathfrak{Op}^A(h) \in \mathscr{B}[\mathcal{S}(\mathbb{R}^d)] \cap \mathscr{B}[\mathcal{S}'(\mathbb{R}^d)]$. We interpret it as a linear operator in $\mathcal{S}'(\mathbb{R}^d)$ and set $H(A, 0) := \mathfrak{Op}^A(h)|_{\mathsf{D}}$.

Now, by applying $\mathfrak{Op}^A$ to (7.3.1) we get

$$\{H(A, 0) - z\mathbf{1}\}\mathfrak{Op}^A[\Phi_h^B(r_z)] = \mathbf{1}$$

and

$$\mathfrak{Op}^A[\Phi_h^B(r_z)]\{\mathfrak{Op}^A(h) - z\mathbf{1}_{\mathcal{S}(\mathbb{R}^d)}\} = \mathbf{1}_{\mathcal{S}(\mathbb{R}^d)}.$$

The first identity shows that $H(A, 0)\mathsf{D} \subset \mathcal{H}$. Straightforwardly it is hermitian. The second equality implies that $\mathcal{S}(\mathbb{R}^d) \subset \mathsf{D}$ and thus D is dense in $\mathcal{H}$. By the first equality above the ranges of $H(A, 0) \pm i$ both coincide with $\mathcal{H}$. Thus, by a fundamental criterion of self-adjointness, $H(A, 0)$ is self-adjoint.

By construction, $\{\mathfrak{Op}^A[\Phi_h^B(r_z)] \mid z \in \mathbb{C}\setminus\mathbb{R}\}$ is the resolvent family of $H(A, 0)$, which is therefore affiliated to $\mathfrak{Op}^A\big(\mathfrak{C}_{\mathscr{C}}^B\big)$.

Then we define the standard operator sum $H(A, V) := H(A, 0) + V : \mathsf{D} \to \mathcal{H}$. Using the second resolvent equation and the Neumann series the conclusion of the Corollary follows easily using [MPR05, Prop. 2.6]. A different proof could start from the result of Corollary 7.3.3. □

We can now present several technical lemmas which have already been used in the previous proofs.

Lemma 7.4.1. *Assume that the components of the magnetic field B belong to $\mathscr{C} \cap BC^\infty(\mathbb{R}^d)$. Then γ^B, defined in* (7.4.2), *belongs to $C^\infty_{pol}(\mathbb{R}^d \times \mathbb{R}^d; \mathscr{C})$, or more precisely:*

(a) *for each $x, y \in \mathbb{R}^d$, $\gamma^B(\cdot; x, y) \in \mathscr{C}$,*

(b) *for each $\alpha, \beta \in \mathbb{N}^d$, there exist $c > 0$, $s_1 \geq 0$ and $s_2 \geq 0$ such that for all $q, x, y \in \mathbb{R}^d$:*

$$\big|\partial_x^\alpha \partial_y^\beta \gamma^B(q; x, y)\big| \leq c\,\langle x\rangle^{s_1}\,\langle y\rangle^{s_2}.$$

Proof. We use the explicit parameterized form of γ^B

$$\gamma^B(q;x,y) = \exp\Big\{ -i \sum_{j,k=1}^{d} x_j\, y_k \int_0^1 \Big[\int_0^1 s\, B_{jk}\big(q - \tfrac{1}{2}x - \tfrac{1}{2}y + sx + st(y-x)\big)\mathrm{d}s\Big]\mathrm{d}t\ \Big\}. \tag{7.4.10}$$

A careful examination of (7.4.10) leads directly to the results (a) and (b). See also the proof of Lemma 4.2 in [MPR05]. □

For the next statement, recall that $F_{a,j}(\cdot;\cdot,\cdot)$ has been introduced in (7.4.3).

Lemma 7.4.2. *For each $j \in \{1,\dots,d\}$, each $\alpha,\beta,\gamma \in \mathbb{N}^d$ and each $\mu > \max\{1,s\}$ there exists $c > 0$ such that*

$$\big|\partial_p^\alpha\, \partial_k^\beta\, \partial_l^\gamma\, F_{a,j}(p;k,l)\big| \le c\, a^{-1/\mu}\, \langle p\rangle^{s/\mu-1-|\alpha|}\, \langle k\rangle^{s}\, \langle l\rangle^{2s} \tag{7.4.11}$$

for all $p,k,l \in \hat{\mathbb{R}}^d$ and $a \ge -\inf h + 1$.

Proof. It is enough to show that the expression

$$\sup_{t\in[0,1]} \Big|\partial_p^\alpha\, \partial_k^\beta\, \partial_l^\gamma \big[(l_j - k_j)\ (\partial_j h)\big(p + (t-1)l - tk\big)\ h_a^{-1}(p-l)\big]\Big| \tag{7.4.12}$$

is dominated by the r.h.s. term of (7.4.11) with a constant c not depending on p, k, l and a.

It is easy to see that for any $\delta \in \mathbb{N}^d$, we have $\partial^\delta h_a^{-1} = h_a^{-1}\, u_{a,\delta}$, where $u_{a,\delta} \in S^{-|\delta|}(\hat{\mathbb{R}}^d)$ uniformly in a. By using this, the Leibnitz formula and the inequality $\langle x+y\rangle \le \sqrt{2}\langle x\rangle\langle y\rangle$, it follows straightforwardly that (7.4.12) is dominated by

$$c_1 h_a^{-1}(p-l)\langle p\rangle^{s-1-|\alpha|}\langle k\rangle^{s}\langle l\rangle^{s}$$

for some $c_1 > 0$ independent of p, k, l and a. Furthermore, by using the ellipticity of h, we see that there exist $c_2 > 0$ and $c_3 > 0$ independent of p,l and a such that $h_a^{-1}(p-l) \le c_2\, \langle l\rangle^{s}[a + c_3\, \langle p\rangle^{s}]^{-1}$ for all $p,l \in \hat{\mathbb{R}}^d$. The final step consists in taking into account the inequality $a + c_3\langle p\rangle^{s} \ge \mu^{1/\mu}\, (\nu c_3)^{1/\nu}\, a^{1/\mu}\, \langle p\rangle^{s/\nu}$, valid for any $\mu \ge 1$, $\nu \ge 1$ with $\mu^{-1} + \nu^{-1} = 1$. □

In order to state the next lemma in its full generality, we need the definition:

Definition 7.4.3. *For $s \in \mathbb{R}$, $S^s(\hat{\mathbb{R}}^d;\mathscr{C})$ denotes the set of all functions $f : \mathbb{R}^d \times \hat{\mathbb{R}}^d \to \mathscr{C}$ that satisfy:*

(i) *$f(\cdot;p) \in \mathscr{C}$ for all $p \in \mathbb{R}^d$,*

(ii) *$f(q;\cdot) \in C^\infty(\hat{\mathbb{R}}^d)$, $\forall q \in \mathbb{R}^d$, and for each $\alpha \in \mathbb{N}^d$*

$$\sup_{q\in\mathbb{R}^d} \|f(q;\cdot)\|_{s,\alpha} := \sup_{q\in\mathbb{R}^d}\ \sup_{p\in\hat{\mathbb{R}}^d}\ \big[\langle p\rangle^{-s+|\alpha|}\, |\partial_p^\alpha f(q;p)|\big] < \infty\ .$$

It is easily seen that the algebraic tensor product $\mathscr{C} \odot S^s(\hat{\mathbb{R}}^d)$ is contained in $S^s(\hat{\mathbb{R}}^d;\mathscr{C})$.

Lemma 7.4.4. *Let f be an element of $S^t(\hat{\mathbb{R}}^d;\mathscr{C})$ with $t < 0$. Then its partial Fourier transform $\mathfrak{F}^{-1}f$ is an element of $L^1(\mathbb{R}^d;\mathscr{C})$ that satisfies for a suitable large integer m*

$$\|\mathfrak{F}^{-1}f\|_{L^1(\mathbb{R}^d;\mathscr{C})} \leq c \max_{|\alpha|\leq m} \sup_{q\in\mathbb{R}^d} \|f(q;\cdot)\|_{t,\alpha} \,. \tag{7.4.13}$$

Proof. This is a straightforward adaptation of the proof of [ABG96, Prop. 1.3.3] (see also [ABG96, Prop. 1.3.6]). We decided to present it in order to put into evidence the explicit bound (7.4.13). Actually, the arguments needed to control the behavior in the variable q are easy and we leave them to the reader; we take simply $f \in S^t(\hat{\mathbb{R}}^d)$.

Since the case $t \leq -d$ is rather simple, we shall concentrate on the more difficult one: $-d < t < 0$. Let us first choose a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$ that is 1 in a neighbourhood of 0. One has the estimates (with $\mathcal{F}$ the Fourier transform but without the constant factor):

$$\begin{aligned}
&\|(1-\chi)\mathcal{F}^{-1}f\|_{L^1} \;\leq\; C\sum_{|\alpha|=m} \||Q|^{-2m}(1-\chi)\mathcal{F}^{-1}(\partial^{2\alpha}f)\|_{L^1}\\
\leq\;& C\Big(\int_{\mathbb{R}^d}\big(1-\chi(x)\big)^2|x|^{-4m}\,\mathrm{d}x\Big)^{1/2}\sum_{|\alpha|=m}\|\partial^{2\alpha}f\|_{L^2}\\
\leq\;& C'\Big(\int_{\mathbb{R}^d}\big(1-\chi(x)\big)^2|x|^{-4m}\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\hat{\mathbb{R}}^d}\langle p\rangle^{2(t-2m)}\,\mathrm{d}p\Big)^{1/2}\max_{|\alpha|=2m}\|f\|_{t,\alpha}\,,
\end{aligned}$$

where we take $m \in \mathbb{N}$ with $4m > d$ to make the integrals convergent.

We study now the behavior of $\mathcal{F}^{-1}f$ near the origin, a more difficult matter. Let us fix a second cutoff function $\varphi \in C^\infty(\hat{\mathbb{R}}^d)$ such that $0 \leq \varphi \leq 1$, $\varphi(p) = 0$ for $|p| \leq 1$ and $\varphi(p) = 1$ for $|p| \geq 2$. For $b > 0$ we set $\varphi_b(p) := \varphi(bp)$. We have:

$$\big|\big\{\mathcal{F}^{-1}\big((1-\varphi_b)f\big)\big\}(y)\big| \;\leq\; \int_{|p|\leq 2/b}|f(p)|\,\mathrm{d}p \;\leq\; \|f\|_{t,0}\int_{|p|<2/b}|p|^t\,\mathrm{d}p \leq C\,\|f\|_{t,0}\,b^{-d-t}.$$

Moreover, if $m \in 2\mathbb{N}$ with $m \geq d+1$, then one has:

$$\begin{aligned}
&|y|^m|[\mathcal{F}^{-1}(\varphi_b f)](y)| \;\leq\; C\sum_{|\alpha|=m}\big|\big[\mathcal{F}^{-1}\big(\partial^\alpha(\varphi_b f)\big)\big](y)\big|\\
\leq\;& C\sum_{|\alpha|=m}\sum_{\beta\leq\alpha}C_\alpha^\beta\, b^{|\alpha-\beta|}\int_{\hat{\mathbb{R}}^d}|(\partial^{\alpha-\beta}\varphi)(bp)|\;|(\partial^\beta f)(p)|\,\mathrm{d}p\\
\leq\;& C'\max_{|\alpha|\leq m}\|f\|_{t,\alpha}\Big\{\int_{|p|\geq 1/b}|p|^{t-m}\,\mathrm{d}p+\sum_{|\beta|<m}b^{m-|\beta|}\int_{1/b<|p|<2/b}|p|^{t-|\beta|}\,\mathrm{d}p\Big\}\\
=\;& C''\max_{|\alpha|\leq m}\|f\|_{t,\alpha}\,b^{m-d-t}.
\end{aligned}$$

By fixing $b := |y|$, we get $|[\mathcal{F}^{-1}(\varphi_{|y|} f)](y)| \leq C'' \max_{|\alpha| \leq m} \|f\|_{t,\alpha} \, |y|^{-d-t}$. The singularity at the origin is integrable, and putting all the inequalities together we obtain (7.4.13). $\square$