Chapter 4

Schrödinger operators and essential spectrum

The aim of this chapter is to show how crossed product C^*-algebras can be used for the computation of some spectral information on self-adjoint operators. These operators appeared naturally in the context of quantum mechanics, but then their investigations has been developed on a pure mathematical level. For simplicity, all the following considerations will be based on the group $\mathbb{R}^d$, but with the content of the previous chapters these investigations could be made on an arbitrary locally compact abelian group. This natural generalization should hold mutatis mutandis, and it is certainly a very useful exercise to check this statement (note that the main difficulties come from the constants and from some historical conventions).

4.1 Multiplication and convolution operators

In this section, we introduce two natural classes of operators on $\mathbb{R}^d$. This material is standard and can be found for example in the books [Amr09] and [Tes09]. We start by considering multiplication operators on the Hilbert space $L^2(\mathbb{R}^d)$.

For any measurable complex function φ on $\mathbb{R}^d$ let us define the *multiplication operator* $\varphi(X)$ on $\mathcal{H} := L^2(\mathbb{R}^d)$ by

$$[\varphi(X)f](x) = \varphi(x)f(x) \qquad \forall x \in \mathbb{R}^d$$

for any

$$f \in \mathcal{D}\big(\varphi(X)\big) := \big\{g \in \mathcal{H} \mid \int_{\mathbb{R}^d} |\varphi(x)|^2 |g(x)|^2 \mathrm{d}x < \infty\big\}.$$

Clearly, the properties of this operator depend on the function φ. More precisely:

Lemma 4.1.1. *Let $\varphi(X)$ be the multiplication operator on $\mathcal{H}$. Then $\varphi(X)$ belongs to $\mathscr{B}(\mathcal{H})$ if and only if $\varphi \in L^\infty(\mathbb{R}^d)$, and in this case $\|\varphi(X)\| = \|\varphi\|_\infty$.*

Proof. One has

$$\|\varphi(X)f\|^2 = \int_{\mathbb{R}^d} |\varphi(x)|^2 |f(x)|^2 \mathrm{d}x \leq \|\varphi\|_\infty^2 \int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x = \|\varphi\|_\infty^2 \|f\|^2.$$

Thus, if $\varphi \in L^\infty(\mathbb{R}^d)$, then $\mathcal{D}\big(\varphi(X)\big) = \mathcal{H}$ and $\|\varphi(X)\| \leq \|\varphi\|_\infty$.

Now, assume that $\varphi \notin L^\infty(\mathbb{R}^d)$. It means that for any $n \in \mathbb{N}$ there exists a measurable set $W_n \subset \mathbb{R}^d$ with $0 < |W_n| < \infty$ such that $|\varphi(x)| > n$ for any $x \in W_n$. We then set $f_n = \chi_{W_n}$ and observe that $f_n \in \mathcal{H}$ with $\|f_n\|^2 = |W_n|$ and that

$$\|\varphi(X)f_n\|^2 = \int_{\mathbb{R}^d} |\varphi(x)|^2 |f_n(x)|^2 \mathrm{d}x = \int_{W_n} |\varphi(x)|^2 dx > n^2 \|f_n\|^2,$$

from which one infers that $\|\varphi(X)f_n\|/\|f_n\| > n$. Since n is arbitrary, the operator $\varphi(X)$ can not be bounded.

Let us finally show that if $\varphi \in L^\infty(\mathbb{R}^d)$, then $\|\varphi(X)\| \geq \|\varphi\|_\infty$. Indeed, for any $\varepsilon > 0$, there exists a measurable set $W_\varepsilon \subset \mathbb{R}^d$ with $0 < |W_\varepsilon| < \infty$ such that $|\varphi(x)| > \|\varphi\|_\infty - \varepsilon$ for any $x \in W_\varepsilon$. Again by setting $f_\varepsilon = \chi_{W_\varepsilon}$ one gets that $\|\varphi(X)f_\varepsilon\|/\|f_\varepsilon\| > \|\varphi\|_\infty - \varepsilon$, from which one deduces the required inequality. □

If $\varphi \in L^\infty(\mathbb{R}^d)$, one easily observes that $\varphi(X)^* = \overline{\varphi}(X)$, and thus $\varphi(X)$ is self-adjoint if and only if φ is a real function. If φ is real but does not belong to $L^\infty(\mathbb{R}^d)$, one can show that the pair $\big(\varphi(X), \mathcal{D}\big(\varphi(X)\big)\big)$ defines a self-adjoint operator if $\mathcal{D}\big(\varphi(X)\big)$ is dense in $\mathcal{H}$. In particular, if $\varphi \in C(\mathbb{R}^d)$ or if $|\varphi|$ is polynomially bounded, then the mentioned operator is self-adjoint, see [Amr09, Prop. 2.29]. For example, for any $j \in \{1, \ldots, d\}$ the operator X_j defined by $[X_j f](x) = x_j f(x)$ is a self-adjoint operator with domain $\mathcal{D}(X_j)$. Note that the d-tuple $(X_1, \ldots, X_d)$ is often referred to as the *position operator* in $L^2(\mathbb{R}^d)$. More generally, for any $\alpha \in \mathbb{N}^d$ one also sets

$$X^\alpha = X_1^{\alpha_1} \ldots X_d^{\alpha_d}$$

and this expression defines a self-adjoint operator on its natural domain. Other useful multiplication operators are defined for any $s > 0$ by the functions

$$\mathbb{R}^d \ni x \mapsto \langle x \rangle^s := \Big(1 + \sum_{j=1}^d x_j^2\Big)^{s/2} \in \mathbb{R}.$$

The corresponding operators $\big(\langle X \rangle^s, \mathcal{H}_s\big)$, with

$$\mathcal{H}_s := \Big\{ f \in \mathcal{H} \mid \langle X \rangle^s f \in \mathcal{H} \Big\} = \Big\{ f \in \mathcal{H} \mid \int_{\mathbb{R}^d} \langle x \rangle^{2s} |f(x)|^2 \mathrm{d}x < \infty \Big\},$$

are again self-adjoint operators on $\mathcal{H}$. Note that one usually calls $\mathcal{H}_s$ *the weighted Hilbert space with weight* s since it is naturally a Hilbert space with the scalar product $\langle f, g \rangle_s := \int_{\mathbb{R}^d} f(x)\overline{g(x)} \langle x \rangle^{2s} \mathrm{d}x$.

Exercise 4.1.2. *For any $\varphi \in C_b(\mathbb{R}^d)$, show that the spectrum of the multiplication operator $\varphi(X)$ coincides with the closure of $\varphi(\mathbb{R}^d)$ in $\mathbb{C}$.*

We shall now introduce a new type of operators on $\mathcal{H}$, but for that purpose we need to recall a few results about the usual Fourier transform[1] on $\mathbb{R}^d$. The Fourier transform $\mathcal{F}$ is defined on any $f \in C_c(\mathbb{R}^d)$ by the formula[2]

$$[\mathcal{F}f](\xi) \equiv \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)\,\mathrm{d}x. \tag{4.1.1}$$

This linear transform maps the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ onto itself, and its inverse is provided by the formula $[\mathcal{F}^{-1}f](x) \equiv \check{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(\xi)\,\mathrm{d}\xi$. In addition, by taking Parseval's identity $\int_{\mathbb{R}^d} |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \mathrm{d}\xi$ into account, one obtains that the Fourier transform extends continuously to a unitary map on $L^2(\mathbb{R}^d)$. We shall keep the same notation $\mathcal{F}$ for this continuous extension, but one must be aware that (4.1.1) is valid only on a restricted set of functions.

Let us use again the multi-index notation and set for any $\alpha \in \mathbb{N}^d$

$$(-i\partial)^\alpha = (-i\partial_1)^{\alpha_1} \dots (-i\partial_d)^{\alpha_d} = (-i)^{|\alpha|} \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$$

with $|\alpha| = \alpha_1 + \dots + \alpha_d$. With this notation at hand, the following relations hold for any $f \in \mathcal{S}(\mathbb{R}^d)$ and any $\alpha \in \mathbb{N}^d$:

$$\mathcal{F}(-i\partial)^\alpha f = X^\alpha \mathcal{F} f,$$

or equivalently $(-i\partial)^\alpha f = \mathcal{F}^* X^\alpha \mathcal{F} f$. Keeping these relations in mind, one defines for any $j \in \{1, \dots, d\}$ the self-adjoint operator $D_j := \mathcal{F}^* X_j \mathcal{F}$ with domain $\mathcal{F}^*\mathcal{D}(X_j)$. Similarly, for any $s > 0$, one also defines the operator $\langle D\rangle^s := \mathcal{F}^*\langle X\rangle^s \mathcal{F}$ with domain

$$\mathcal{H}^s := \big\{f \in \mathcal{H} \mid \langle X\rangle^s \mathcal{F} f \in \mathcal{H}\big\} \equiv \big\{f \in \mathcal{H} \mid \langle X\rangle^s \hat{f} \in \mathcal{H}\big\}.$$

Note that the space $\mathcal{H}^s$ is called *the Sobolev space of order s*, and $(D_1, \dots, D_d)$ is usually called *the momentum operator*[3].

We can now introduce the usual *Laplace operator* $-\Delta$ acting on any $f \in \mathcal{S}(\mathbb{R}^d)$ as

$$-\Delta f = -\sum_{j=1}^d \partial_j^2 f = \sum_{j=1}^d (-i\partial_j)^2 f = \sum_{j=1}^d D_j^2 f. \tag{4.1.2}$$

[1] In the more general framework of arbitrary locally compact abelian group, the Fourier transform has been defined at the end of Section 3.2. The constants are chosen here such that the Fourier transform extends to a unitary map on $L^2(\mathbb{R}^d)$.

[2] Even if the group $\mathbb{R}^d$ is identified with its dual group, we will keep the notation ξ for points of its dual group.

[3] In physics textbooks, the position operator is often denoted by $(Q_1, \dots, Q_d)$ while $(P_1, \dots, P_d)$ is used for the momentum operator.

This operator admits a self-adjoint extension with domain $\mathcal{H}^2$, *i.e.* $\big(-\Delta,\mathcal{H}^2\big)$ is a self-adjoint operator in $\mathcal{H}$. However, let us stress that the expression (4.1.2) is not valid (pointwise) on all the elements of the domain $\mathcal{H}^2$. On the other hand, one has $-\Delta=\mathcal{F}^*X^2\mathcal{F}$, with $X^2=\sum_{j=1}^d X_j^2$, from which one easily infers that $\sigma(-\Delta)=[0,\infty)$. Indeed, this follows from the content of Exercise 4.1.2 together with the invariance of the spectrum through the conjugation by a unitary operator.

More generally, for any measurable function φ on $\mathbb{R}^d$ one sets $\varphi(D):=\mathcal{F}^*\varphi(X)\mathcal{F}$, with domain $\mathcal{D}\big(\varphi(D)\big)=\big\{f\in\mathcal{H}\mid \hat{f}\in\mathcal{D}\big(\varphi(X)\big)\big\}$, and as before this operator is self-adjoint if this domain is dense in $\mathcal{H}$, as for example for a continuous function φ or for a polynomially bounded function φ. Then, if one defines the convolution of two (suitable) functions on $\mathbb{R}^d$ by the formula

$$[k*f](x)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}k(y)f(x-y)\,\mathrm{d}y$$

and if one takes the equality $\check{g}*f=\mathcal{F}^*(g\hat{f})$ into account, one infers that the operator $\varphi(D)$ corresponds to *a convolution operator*, namely

$$\varphi(D)f=\check{\varphi}*f. \tag{4.1.3}$$

Obviously, the meaning of such an equality depends on the class of functions f and g considered.

Exercise 4.1.3. *Show that the following relations hold on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$: $[iX_j,X_k]=\mathbf{0}=[D_j,D_k]$ for any $j,k\in\{1,\dots,d\}$ while $[iD_j,X_k]=\mathbf{1}\delta_{jk}$.*

4.2 Schrödinger operators

In this section, we introduce the main operator we want to investigate.

First of all, let $h:\mathbb{R}^d\to\mathbb{R}$ be a continuous real function which diverges at infinity. Equivalently, we assume that h satisfies $(h-z)^{-1}\in C_0(\mathbb{R}^d)$ for some $z\in\mathbb{C}\setminus\mathbb{R}$. The corresponding convolution operator $h(D)$, defined by $\mathcal{F}^*h(X)\mathcal{F}$, is a self-adjoint operator with domain $\mathcal{F}^*\mathcal{D}\big(h(X)\big)$. Clearly, the spectrum of such an operator is equal to the closure of $h(\mathbb{R}^d)$ in $\mathbb{R}$.

Some examples of such a function h which are often considered in the literature are the functions defined by $h(\xi)=\xi^2$, $h(\xi)=|\xi|$ or $h(\xi)=\sqrt{1+\xi^2}-1$. In these cases, the operator $h(D)=-\Delta$ corresponds to *the free Laplace operator*, the operator $h(D)=|D|$ is *the relativistic Schrödinger operator without mass*, while the operator $h(D)=\sqrt{1-\Delta}-1$ corresponds to *the relativistic Schrödinger operator with mass.* In these three cases, one has $\sigma\big(h(D)\big)=\sigma_{ac}\big(h(D)\big)=[0,\infty)$ while $\sigma_{sc}\big(h(D)\big)=\sigma_p\big(h(D)\big)=\emptyset$.

Let us now perturb this operator $h(D)$ with a multiplication operator $V(X)$. If the measurable function $V:\mathbb{R}^d\to\mathbb{R}$ is not essentially bounded, then the operator $h(D)+V(X)$ can only be defined on the intersection of the two domains, and checking that there exists a self-adjoint extension of this operator is not always an easy task. On the other hand, if one assumes that $V \in L^\infty(\mathbb{R}^d)$, then we can define the operator

$$H := h(D) + V(X) \qquad \text{with} \quad \mathcal{D}(H) = \mathcal{D}\big(h(D)\big) \tag{4.2.1}$$

and this operator is self-adjoint.

A lot of investigations have been performed on such an operator H when V vanishes at infinity, in a suitable sense. On the other hand, much less is known on this operator when the multiplication operator $V(X)$, also called *the potential*, has an anisotropic behavior. The main idea of the approach presented here is to encode the anisotropy in an algebra $\mathscr{C}$ of functions on $\mathbb{R}^d$. Then, if the potential belongs to this algebra of functions, we can show that the operator H itself belongs to the crossed product algebra. More explanations about this construction are provided in the next section.

4.3 Affiliation

The main ideas of this section are borrowed from [Măn02]. Some other references using similar ideas are [GI02, AMP02, GI06, DG13, Măn013]. From now on, we consider an algebra of functions on $\mathbb{R}^d$ satisfying the following assumptions:

Assumption 4.3.1. *$\mathscr{C}$ is a unital C^*-subalgebra of $BC_u(\mathbb{R}^d)$ which is $\mathbb{R}^d$-invariant and which contains the subalgebra $C_0(\mathbb{R}^n)$.*

Recall that this algebra is $\mathbb{R}^d$-invariant if whenever $\varphi \in \mathscr{C}$ and $x \in \mathbb{R}^d$, then $\theta_x(\varphi) := \varphi(\cdot - x) \in \mathscr{C}$. As a consequence of Theorem 2.4.15, there exists a compact space Ω such that $\mathscr{C}$ is isometrically $*$-isomorphic to $C(\Omega)$. In addition, note that from the requirement $C_0(\mathbb{R}^d) \subset \mathscr{C}$ one infers that Ω is *a compactification of* $\mathbb{R}^d$ ($\Leftrightarrow$ a compact space in which $\mathbb{R}^d$ is dense). Indeed, each point $x \in \mathbb{R}^d$ defines a distinct element of the character space Ω by setting $x \to \delta_x$ where δ_x is the evaluation at x, *i.e.* $\delta_x(\varphi) := \varphi(x)$ for any $\varphi \in \mathscr{C}$. Finally, one also observes that the action of $\mathbb{R}^d$ continuously extends to an action on Ω defined by the formula:

$$[\theta_x(\tau)](\varphi) = \tau\big(\theta_x(\varphi)\big), \tag{4.3.1}$$

for any $\varphi \in \mathscr{C}$, $x \in \mathbb{R}^d$ and $\tau \in \Omega$. Note that we use the same symbol for the action of $\mathbb{R}^d$ on itself and for its extension on Ω. In summary, the Assumptions 4.3.1 imply that the triple $\big(C(\Omega), \mathbb{R}^d, \theta\big)$ defines a C^*-dynamical system, see also Example 3.3.2 and Exercise 3.3.3.

Exercise 4.3.2. *Find a unital C^*-subalgebra of $BC_u(\mathbb{R}^n)$ which is $\mathbb{R}^d$-invariant but for which the space Ω is not a compactification of $\mathbb{R}^d$.*

Let us now construct a covariant representation of this dynamical system in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$. First of all, the algebra $C(\Omega)$ is faithfully represented in $\mathscr{B}(\mathcal{H})$ by operators of multiplication. Indeed, if one defines the $*$-homomorphism π by

$$C(\Omega) \cong \mathscr{C} \ni \varphi \mapsto \pi(\varphi) := \varphi(X) \in \mathscr{B}(\mathcal{H}),$$

then $\pi : C(\Omega) \to \mathscr{B}(\mathcal{H})$ is faithful and non-degenerate. In addition, let us consider the unitary representation of the group $\mathbb{R}^d$ on $\mathcal{H}$, namely $\{U_x\}_{x\in\mathbb{R}^d}$ given by $[U_x f](y) = f(y-x)$ for any $f \in \mathcal{H}$. With this definition, the following compatibility condition holds for any $\varphi \in \mathscr{C}$

$$\pi\big(\theta_x(\varphi)\big) = \pi\big(\varphi(\cdot - x)\big) = \varphi(X-x) = U_x\varphi(X)U_x^* = U_x\pi(\varphi)U_x^*. \qquad (4.3.2)$$

As a consequence, the triple $(\mathcal{H}, \pi, U)$ defines a covariant representation of the dynamical system $\big(C(\Omega), \mathbb{R}^d, \theta\big)$, and thus a non-degenerate representation of the crossed product algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ in $\mathscr{B}(\mathcal{H})$, by Theorem 3.4.8. This representation corresponds to the integrated representation $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$.

Exercise 4.3.3. *Check that the above operator U_x is equal to the operator $e^{-ix\cdot D}$, where D is the momentum operator introduced in Section 4.1.*

In order to get a better understanding of the C^*-algebra $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$, recall that $C(\Omega)$ is unital, and therefore that $L^1(\mathbb{R}^d) \subset C(\Omega) \rtimes_\theta \mathbb{R}^d$. Thus, by applying the integrated representation $\pi \rtimes U$ defined in (3.4.1) to some $u \in L^1(\mathbb{R}^d)$, one gets

$$[\pi \rtimes U(u)f](x) = \Big[\int_{\mathbb{R}^d} u(y)U_y f\,\mathrm{d}y\Big](x) = \int_{\mathbb{R}^d} u(y)f(x-y)\,\mathrm{d}y = (2\pi)^{d/2}[\hat{u}(D)f](x),$$

where we have taken equation (4.1.3) into account. More generally, by considering products $u \otimes \varphi \subset L^1(\mathbb{R}^d) \odot C(\Omega) \subset L^1\big(\mathbb{R}^d; C(\Omega)\big)$, we get that operators of the form $(2\pi)^{d/2}\varphi(X)\hat{u}(D)$ belong to $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$. Finally, by considering linear combinations, one infers that:

Theorem 4.3.4. *Let $\mathscr{C}$ satisfy Assumption 4.3.1 and let Ω be defined by the Gelfand $*$-isomorphism $\mathscr{C} \cong C(\Omega)$. Then $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ is equal to*

$$\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle := C^*\Big(\varphi(X)v(D) \mid v \in C_0(\mathbb{R}^d) \text{ and } \varphi \in C(\Omega)\Big), \qquad (4.3.3)$$

and the C^*-algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ is isometrically $*$-isomorphic to this algebra.

Proof. By construction, and since $\pi \rtimes U(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ is norm closed, it is quite clear that this algebra and the C^*-algebra defined in the r.h.s. of (4.3.3) are equal. However, it remains to show that the representation $\pi \rtimes U$ of $C(\Omega) \rtimes_\theta \mathbb{R}^d$ is faithful. Such a statement has been proved for example in [GI02, Thm 4.1] and is based on the regular representation introduced in Example 3.3.6. We do not provide the arguments here since we are going to prove a more general result in the context of twisted crossed product C^*-algebras in a forthcoming chapter. □

Remark 4.3.5. *In the previous statement, if one chooses*[4] *$C_0(\mathbb{R}^d)$ for the algebra $\mathscr{C}$, then the resulting C^*-algebra $\langle C_0(\mathbb{R}^d) \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ coincides with C^*-algebra $\mathscr{K}\big(L^2(\mathbb{R}^d)\big)$. It means that the integrated representation $\pi \rtimes U$ provides the $*$-isomorphism already mentioned in Example 3.4.3.*

[4]Obviously, $C_0(\mathbb{R}^d)$ is not unital, but this lack of a unity can easily be taken into account in the previous construction.

We are now in a suitable position for explaining the link between the Schrödinger operator H and the C^*-algebra introduced in (4.3.3).

Lemma 4.3.6. *Let $\mathscr{C}$ satisfy Assumption 4.3.1 and let Ω be defined by the Gelfand $*$-isomorphism $\mathscr{C} \cong C(\Omega)$. Let $h \in C(\mathbb{R}^d;\mathbb{R})$ be diverging at infinity, let $V \in C(\Omega;\mathbb{R})$, and let $H := h(D) + V(X)$. Then, for some $z \in \mathbb{C}\setminus\mathbb{R}$ with $|\Im z|$ large enough, the resolvent $(H-z)^{-1}$ belongs to the C^*-algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$.*

Proof. Let us consider the Neumann series

$$\begin{aligned}(H-z)^{-1} &= \big(h(D) - z + V(X)\big)^{-1} \\ &= \big(h(D)-z\big)^{-1}\Big(\mathbf{1} + V(X)\big(h(D)-z\big)^{-1}\Big)^{-1} \\ &= \big(h(D)-z\big)^{-1}\sum_{n=0}^{\infty}(-1)^n\big[V(X)\big(h(D)-z\big)^{-1}\big]^n,\end{aligned}$$

where we have used the result of Lemma 4.1.1 and suitably chosen z such that

$$\big\|V(X)\big(h(D)-z\big)^{-1}\big\| < 1.$$

Since each term in the series belongs to $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, and since the series converges in norm of $\mathscr{B}(\mathcal{H})$, it follows that the series converges in $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. □

Note that from the resolvent equation (1.6.1), one infers the equalities

$$\begin{aligned}(H-z)^{-1} &= (H-z_0)^{-1}\big(\mathbf{1} + (z-z_0)(H-z_0)^{-1}\big)^{-1} \\ &= \sum_{n=0}^{\infty}(z-z_0)^n\big((H-z_0)^{-1}\big)^{n+1}.\end{aligned}$$

By starting then from the result of the previous lemma and by a approximation argument, one deduces that if $(H-z_0)^{-1} \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for some $z_0 \in \mathbb{C}\setminus\mathbb{R}$, then $(H-z)^{-1} \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. By a density argument, it even follows that $\varphi(H) \in \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ for any $\varphi \in C_0(\mathbb{R})$, and the operator $\varphi(H)$ corresponds to the one also mentioned in Definition 1.7.9. Thus, H defines a $*$-homomorphism from $C_0(\mathbb{R})$ to $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. More generally, one sets:

Definition 4.3.7. *(i)* An observable affiliated to a C^*-algebra $\mathfrak{C}$ *is a $*$-homomorphism $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.*

(ii) The spectrum $\sigma(\Phi)$ of an observable Φ *consists in the set of $\lambda \in \mathbb{R}$ such that $\Phi(\varphi) \neq \mathbf{0}$ whenever $\varphi \in C_0(\mathbb{R})$ and $\varphi(\lambda) \neq 0$.*

Let us stress that for the previous definition of an observable, the C^*-algebra $\mathfrak{C}$ does not need to be represented in a Hilbert space. On the other hand, with the observation made just before the definition, one observes that if H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, and if $\mathfrak{C}$ is a C^*-subalgebra of $\mathscr{B}(\mathcal{H})$ with $(H-z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C}\setminus\mathbb{R}$, then H defines an observable affiliated to $\mathfrak{C}$, which we denote by Φ^H (in this case one has $\Phi^H(\varphi) = \varphi(H)$).

Exercise 4.3.8. *In the framework of the previous paragraph, show that $\sigma\big(\Phi^H\big) = \sigma(H)$.*

4.4 $\mathfrak{J}$-essential spectrum

Let us consider a C^*-algebra $\mathfrak{C}$ and one ideal $\mathfrak{J}$ in $\mathfrak{C}$ (in this section ideals of C^*-algebras will always be considered closed and self-adjoint). By Corollary 2.5.6, the quotient algebra $\mathfrak{C}/\mathfrak{J}$ is a C^*-algebra, and let us denote by $q : \mathfrak{C} \to \mathfrak{C}/\mathfrak{J}$ the quotient $*$-homomorphism. Then, if Φ is an observable affiliated to $\mathfrak{C}$, the composed map $q \circ \Phi : C_0(\mathbb{R}) \to \mathfrak{C}/\mathfrak{J}$ defines an observable affiliated to the quotient algebra. In this setting, one has:

Definition 4.4.1. *Let $\mathfrak{C}$ be a C^*-algebra, with $\mathfrak{J}$ an ideal in $\mathfrak{C}$, and let Φ be an observable affiliated to $\mathfrak{C}$. The spectrum $\sigma(q \circ \Phi)$ of the observable $q \circ \Phi$ is called* the $\mathfrak{J}$-essential spectrum *of Φ and will be denoted by $\sigma^{\mathfrak{J}}(\Phi)$,* i.e. $\sigma^{\mathfrak{J}}(\Phi) = \sigma(q \circ \Phi)$.

Exercise 4.4.2. *In the framework of the previous definition, show that $\lambda \in \sigma^{\mathfrak{J}}(\Phi)$ if and only if $\Phi(\varphi) \notin \mathfrak{J}$ whenever $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) \neq 0$.*

To motivate the introduction of this notion of $\mathfrak{J}$-essential spectrum, let us derive the original result in this setting:

Lemma 4.4.3. *Let $\mathcal{H}$ be a Hilbert space, and H be a self-adjoint operator on $\mathcal{H}$. Then the following equality holds:*

$$\sigma_{ess}(H) = \sigma^{\mathscr{K}(\mathcal{H})}\big(\Phi^H\big),$$

or in other words, the essential spectrum of H can be computed by considering the $\mathscr{K}(\mathcal{H})$-essential spectrum of the corresponding observable affiliated to $\mathscr{B}(\mathcal{H})$.

Proof. From the definition of the essential spectrum provided in Definition 1.7.17, it is easily observed that $\lambda \in \sigma_{ess}(H)$ if and only if $E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H}$ is infinite dimensional for any $\delta > 0$, where $E(\cdot)$ denotes the spectral measure associated with the self-adjoint operator H, see Section 1.7.2. This property in then equivalent to the fact that if $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) > 0$, the corresponding operator $\Phi^H(\varphi) = \varphi(H) \notin \mathscr{K}(\mathcal{H})$. Indeed:

$\Leftarrow$: let $\delta > 0$ and choose $\varphi \in C_c\big((\lambda-\delta, \lambda+\delta)\big)$ with $\varphi(\lambda) > 0$. By assumption $\varphi(H) \notin \mathscr{K}(\mathcal{H})$, and therefore $E\big((\lambda-\delta, \lambda+\delta)\big) \notin \mathscr{K}(\mathcal{H})$ since otherwise one would have $\varphi(H) = \varphi(H)E\big((\lambda-\delta, \lambda+\delta)\big) \in \mathscr{K}(\mathcal{H})$.

$\Rightarrow$: By absurd let us assume that there exists $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) > 0$ such that $\varphi(H) \in \mathscr{K}(\mathcal{H})$. Therefore, for any $\varepsilon > 0$ with $\varphi(\lambda)/2 > \varepsilon$ there exist $\{g_j, h_j\}_{j=1}^n \subset \mathcal{H}$ such that $\|\varphi(H) - A_n\| < \varepsilon$, see equation (1.3.1) for the definition of A_n. Then, let us choose $\delta > 0$ such that $\varphi(\lambda') > \varphi(\lambda) - \varepsilon$ for any $\lambda' \in (\lambda-\delta, \lambda+\delta)$. By assumption, the subspace $E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H}$ is infinite dimensional, and so is the subspace

$$E\big((\lambda-\delta, \lambda+\delta)\big)\mathcal{H} \cap \mathsf{Vect}\left(\{g_j, h_j \mid j \in \{1, \dots, n\}\}\right)^{\perp}.$$

It finally follows from Proposition 1.7.4.(iii) for any f in the above set one has

$$\|\varphi(H)f\|^2 = \int_{\lambda-\delta}^{\lambda+\delta} |\varphi(\lambda')|^2 m_f(\mathrm{d}\lambda') > (\varphi(\lambda)-\varepsilon)^2 \int_{\lambda-\delta}^{\lambda+\delta} m_f(\mathrm{d}\lambda') = (\varphi(\lambda)-\varepsilon)^2\|f\|^2,$$

implying that $\|\varphi(H)f\| > (\varphi(\lambda)-\varepsilon)\|f\| > \varphi(\lambda)\|f\|/2 > \varepsilon\|f\|$. However, this estimate contradicts the initial assumption which stated that

$$\|\varphi(H)f\| = \big\|\big(\varphi(H)-A_n\big)f\big\| < \varepsilon\|f\|.$$

□

Now, if $\mathfrak{J}$ is an ideal in a C^*-algebra $\mathfrak{C}$, the computation of the $\mathfrak{J}$-essential spectrum of an observable Φ affiliated to $\mathfrak{C}$ can sometimes be eased by the existence of a larger family of ideals $\mathfrak{J}_i$ in $\mathfrak{C}$ which satisfy $\cap_i\mathfrak{J}_i = \mathfrak{J}$. Our interest in such a family is that the quotient algebras $\mathfrak{C}/\mathfrak{J}_i$ might be more easily understandable than the quotient $\mathfrak{C}/\mathfrak{J}$. Note that in this framework we shall denote by q the quotient map $\mathfrak{C}\to\mathfrak{C}/\mathfrak{J}$ and by q_i the quotient map $\mathfrak{C}\to\mathfrak{C}/\mathfrak{J}_i$. Our next aim is to show that with such a construction, the spectral properties are preserved. Note that we shall use the notation $\hookrightarrow$ for injective $*$-homomorphisms.

Proposition 4.4.4. *Let $\mathfrak{C}$ be a C^*-algebra, and $\mathfrak{J},\mathfrak{J}_i$ be ideals in $\mathfrak{C}$ satisfying $\cap_i\mathfrak{J}_i = \mathfrak{J}$.*

(i) There is a canonical injective $*$-homomorphism $\mathfrak{C}/\mathfrak{J}\hookrightarrow \Pi_i\ \mathfrak{C}/\mathfrak{J}_i$,

(ii) If Φ is an observable affiliated to $\mathfrak{C}$, and if one sets $\Phi_i := q_i\circ\Phi$ for the observable affiliated to $\mathfrak{C}/\mathfrak{J}_i$, then one has

$$\sigma^{\mathfrak{J}}(\Phi) = \overline{\cup_i\ \sigma(\Phi_i)} \tag{4.4.1}$$

Proof. With the notation introduced before the statement, one has that the kernel of q_i is $\mathfrak{J}_i$. Thus, the kernel of $(q_i)_i : \mathfrak{C}\to\Pi_i\ \mathfrak{C}/\mathfrak{J}_i$ is $\cap_i\mathfrak{J}_i = \mathfrak{J}$.

By definition, one has

$$\begin{aligned}\sigma^{\mathfrak{J}}(\Phi) &= \sigma(q\circ\Phi)\\ &= \big\{\lambda\in\mathbb{R}\mid q\big(\Phi(\varphi)\big)\neq\mathbf{0}\ \forall\varphi\in C_0(\mathbb{R})\text{ with }\varphi(\lambda)\neq 0\big\}\\ &= \overline{\cup_i\big\{\lambda\in\mathbb{R}\mid q_i\big(\Phi(\varphi)\big)\neq\mathbf{0}\ \forall\varphi\in C_0(\mathbb{R})\text{ with }\varphi(\lambda)\neq 0\big\}}\\ &= \overline{\cup_i\sigma(\Phi_i)}.\end{aligned}$$

Alternatively, we can use that for any $\varphi\in C_0(\mathbb{R})$ one has

$$\sigma\big(q\circ\Phi(\varphi)\big) = \sigma\big[q\big(\Phi(\varphi)\big)\big] = \sigma\big[\Pi_i q_i\big(\Phi(\varphi)\big)\big] = \overline{\cup_i\sigma\big(\Phi_i(\varphi)\big)}$$

where we have used that the spectrum is invariant under an injective $*$-homomorphism[5] and that the spectrum of an operator belonging to a direct product is the closure of the union of the spectrum of its components. □

[5]Indeed, if $\mathscr{C}$, $\mathscr{Q}$ are C^*-algebras and if $\varphi:\mathscr{C}\to\mathscr{Q}$ is an injective $*$-homomorphism, it follows from Corollary 2.5.8 that $\mathscr{C}$ and $\varphi(\mathscr{C})\subset\mathscr{Q}$ are isometrically $*$-isomorphic, and thus computing the spectrum of $A\in\mathscr{C}$ or of $\varphi(A)\in\mathscr{Q}$ provides the same result.

Remark 4.4.5. *In the above framework, if $\mathfrak{C} = \mathscr{B}(\mathcal{H})$ and if $\mathfrak{J} = \mathscr{K}(\mathcal{H})$ then the quotient algebra $\mathscr{B}(\mathcal{H})/\mathscr{K}(\mathcal{H})$ is called* the Calkin algebra. *In this situation, there does not exist any other ideal in $\mathscr{B}(\mathcal{H})$, and thus the construction provided in the previous proposition is useless. However, if $\mathfrak{C}$ is a C^*-subalgebra smaller than $\mathscr{B}(\mathcal{H})$ but with $\mathscr{K}(\mathcal{H}) \subset \mathfrak{C}$, then the above construction might provide lots of information, as we shall see in the following section.*

4.5 Orbits and essential spectrum

Our aim in this section is to compute the essential spectrum of the operator $H = h(D)+V(X)$, with $h : \mathbb{R}^d \to \mathbb{R}$ a continuous real function which diverges at infinity, and with $V \in \mathscr{C}$, this C^*-algebra satisfying itself Assumptions 4.3.1. Since by Lemma 4.3.6 the operator H corresponds to an observable affiliated to the C^*-algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ defined in (4.3.3), and since by Remark 4.3.5 we already know that $\mathscr{K}(\mathcal{H}) \subset \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, Proposition 4.4.4 encourages us to find a suitable family of other ideals of $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$ surrounding $\mathscr{K}(\mathcal{H})$ in the sense of that proposition. Thanks to the functoriality of the crossed product, as presented in Corollary 3.5.2, these investigations can be performed quite easily at an abelian level.

Recall first that Ω is a compactification of $\mathbb{R}^d$. In addition, the group $\mathbb{R}^d$ acts continuously on Ω, and we use the notation $\theta_x(\tau)$ for the action of $x \in \mathbb{R}^d$ on $\tau \in \Omega$, see also Exercise 3.3.3. Clearly, $\mathbb{R}^d$ is an open and $\mathbb{R}^d$-invariant subset of Ω, and therefore $C_0(\mathbb{R}^d)$ corresponds to a $\mathbb{R}^d$-invariant ideal of $C(\Omega)$, see the end of Section 3.5. Now, if we denote by $\partial\Omega$ the boundary of Ω ($\Leftrightarrow \Omega \setminus \mathbb{R}^d$), then $\partial\Omega$ is a closed $\mathbb{R}^d$-invariant subset of Ω. By taking into account corollary 3.5.2 we deduce the existence of the following short exact sequence of C^*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}^d) \rtimes_\theta \mathbb{R}^d \longrightarrow C(\Omega) \rtimes_\theta \mathbb{R}^d \longrightarrow C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \longrightarrow 0. \tag{4.5.1}$$

Note that by Theorem 4.3.4 and Remark 4.3.5 we already know faithful representations of the first two algebras in the Hilbert space $\mathcal{H}$. Our aim is thus to obtain a better understanding of the third algebra, by decomposing it into suitable components.

Definition 4.5.1. *Let (Ω, G, θ) be a locally compact transformation group, and let $\tau \in \Omega$.* The orbit $\mathcal{O}_\tau$ of τ *is the set $\{\theta_x(\tau) \mid x \in G\}$, while the quasi-orbit $\mathcal{Q}_\tau$ of τ corresponds to the closure of $\mathcal{O}_\tau$ in Ω.*

Clearly, each orbit and each quasi-orbits are $\mathbb{R}^d$-invariant subsets of Ω. In addition, observe that if $\tau \in \mathbb{R}^d \subset \Omega$, then $\mathcal{O}_\tau$ is a dense orbit in Ω, and therefore $\mathcal{Q}_\tau = \Omega$. On the other hand, if we choose $\tau \in \Omega \setminus \mathbb{R}^d$, then $\mathcal{O}_\tau \subset \partial\Omega$ and $\mathcal{Q}_\tau$ is therefore a closed subset of $\partial\Omega$. Remark however that the set of all quasi-orbits is not a partition of Ω, since quasi-orbits may overlap or there may even be a strict inclusion between them. For that reason, a quasi-orbit is said *maximal* if it is not strictly contained in some other quasi-orbit. On the other hand, note that a subset Ω' of Ω is *minimal* if this set is non-empty, closed and invariant and if no proper subset of Ω' has these three properties. For example, a quasi-orbit is minimal if it does not contain any other proper quasi-orbit. Note that any quasi-orbit contains a minimal one (it may be the quasi-orbit itself).

Exercise 4.5.2. *For any $\tau \in \partial\Omega$ and for any $f \in C(\mathcal{Q}_\tau)$, check that the map*

$$\mathbb{R}^d \ni x \mapsto f\big(\theta_x(\tau)\big) \in \mathbb{C} \tag{4.5.2}$$

is an element of $BC_u(\mathbb{R}^d)$, and that the map $f \mapsto f\big(\theta_\cdot(\tau)\big)$ is injective. Note that from now on and with a slight abuse of notation, we shall always identify $C(\mathcal{Q}_\tau)$ with its realization as a subalgebra of $BC_u(\mathbb{R}^d)$, as prescribed by (4.5.2).

Let us now consider $\{\mathcal{Q}_{\tau_i}\}_i$ a covering of $\partial\Omega$ by quasi-orbits. Clearly, it implies the existence of an injective $*$-homomorphism

$$\varphi : C(\partial\Omega) \ni f \hookrightarrow (f_i)_i \in \Pi_i\, C(\mathcal{Q}_{\tau_i}),$$

where f_i corresponds to the restriction of f to $\mathcal{Q}_{\tau_i}$. Note that this morphism is rarely surjective, but that the following condition holds, namely

$$\lim_{x\to 0}\sup_i \|\theta^i_x(f_i) - f_i\| = \lim_{x\to 0}\sup_i \big\|\theta^i_x(f|_{\mathcal{Q}_{\tau_i}}) - f|_{\mathcal{Q}_{\tau_i}}\big\| = \lim_{x\to 0}\|\theta_x(f) - f\| = 0. \tag{4.5.3}$$

Here θ^i_x denotes the restriction of θ_x to $\mathcal{Q}_{\tau_i}$. In order to keep track of the property (4.5.3), we denote by $\Pi'_i C(\mathcal{Q}_{\tau_i})$ the C^*-subalgebra of $\Pi_i C(\mathcal{Q}_{\tau_i})$ on which this continuity property holds. From these considerations, one infers that $\big(\Pi'_i C(\mathcal{Q}_{\tau_i}), \mathbb{R}^d, \Pi_i\theta^i\big)$ is a C^*-dynamical system and that

$$\varphi' : C(\partial\Omega) \ni f \hookrightarrow (f_i)_i \in \Pi'_i\, C(\mathcal{Q}_{\tau_i})$$

is an equivariant $*$-homomorphism. Then, by the functoriality of the crossed product (see Lemma 3.4.9), one infers that

$$C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \hookrightarrow \Big(\Pi'_i\, C(\mathcal{Q}_{\tau_i})\Big) \rtimes_{\Pi_i\theta^i} \mathbb{R}^d \hookrightarrow \Pi_i\big(C(\mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d\big),$$

where we have taken into account the identification of $C(\mathcal{Q}_{\tau_i})$ with a C^*-subalgebra of $BC_u(\mathbb{R}^d)$ as mentioned in Exercise 4.5.2.

By summarizing our findings, one has obtained that

$$\begin{aligned}
\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \Big/ \mathscr{K}\big(L^2(\mathbb{R}^d)\big) &= \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \Big/ \langle C_0(\mathbb{R}^d) \cdot C_0(\hat{\mathbb{R}}^d)\rangle \\
&\cong \mathscr{C} \rtimes_\theta \mathbb{R}^d \Big/ C_0(\mathbb{R}^d) \rtimes_\theta \mathbb{R}^d \\
&\cong C(\partial\Omega) \rtimes_\theta \mathbb{R}^d \\
&\hookrightarrow \Big(\Pi'_i\, C(\mathcal{Q}_{\tau_i})\Big) \rtimes_{\Pi_i\theta^i} \mathbb{R}^d \\
&\hookrightarrow \Pi_i\,\big(C(\mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d\big) \\
&\cong \Pi_i\,\langle C(\mathcal{Q}_{\tau_i}) \cdot C_0(\hat{\mathbb{R}}^d)\rangle.
\end{aligned} \tag{4.5.4}$$

We shall denote by ι_{ess} the resulting injective $*$-homomorphism.

Remark 4.5.3. *Note that the same result would have been obtained if we had considered the ideals $C_0(\Omega \setminus \mathcal{Q}_{\tau_i})$ of $C(\Omega)$, and observed that $\cap_i\, C_0(\Omega \setminus \mathcal{Q}_{\tau_i}) = C_0(\mathbb{R}^d)$. Then, by identifying in Proposition 4.4.4 the algebra $\mathfrak{C}$ with $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and the ideals $\mathfrak{J}_i$ with $C_0(\Omega \setminus \mathcal{Q}_{\tau_i}) \rtimes_\theta \mathbb{R}^d$, the first statement of this proposition would have coincide with the above result.*

We can now state the main result of this section:

Theorem 4.5.4. *Let $H = h(D) + V(X)$ be the self-adjoint operator defined in Lemma 4.3.6. Let $\{\mathcal{Q}_{\tau_i}\}_i$ be a covering of $\partial\Omega$ by quasi-orbits, and let us set $V_i := V\big(\theta.(\tau_i)\big) \in BC_u(\mathbb{R}^d)$ and $H_i := h(D) + V_i(X)$. Then*

$$\sigma_{ess}(H) = \overline{\cup_i\, \sigma(H_i)}. \tag{4.5.5}$$

Proof. In fact, most of the proof has already been performed before the statement. Indeed, by Lemma 4.3.6 we already know that H defines an observable Φ^H affiliated to the algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$. Then, by keeping track of the form of all the $*$-homomorphisms, we see that ι_{ess} transforms the class modulo $\mathscr{K}(\mathcal{H})$ of the element $V(X)\big(h(D) - z\big)^{-1}$ into $\big(V_i(X)\big(h(D) - z\big)^{-1}\big)_i$. Thus, if q denotes the map

$$q : \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \to \langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle \Big/ \mathscr{K}\big(L^2(\mathbb{R}^d)\big)$$

then by taking the Neumann series into account, one deduces that $\iota_{ess}(q \circ \Phi^H) = \big(\Phi^{H_i}\big)_i$. Finally, since the spectrum is invariant under an injective $*$-homomorphism and since the spectrum of an operator belonging to a direct product is the closure of the union of the spectrum of its components, one directly gets

$$\sigma_{ess}(H) = \sigma(q \circ \Phi^H) = \overline{\cup_i\, \sigma(\Phi^{H_i})} = \overline{\cup_i\, \sigma(H_i)}.$$

□

Note that this result should be compared with the content of Proposition 1.7.18. Note also that such a result holds for more general observables affiliated to the algebra $\langle \mathscr{C} \cdot C_0(\hat{\mathbb{R}}^d)\rangle$, but stronger affiliation criteria are then necessary.

In the publications [Măn02] and [AMP02], highly non-trivial applications of the previous result have been presented. In part of these examples, the index i belongs to a continuum, and the corresponding result could hardly be guessed by constructing Weyl sequences, as introduced in Proposition 1.7.18. On the other hand, let us present a situation which is much more tractable, see [Ric05] for details.

Example 4.5.5 (Cartesian anisotropy)**.** *Let $\mathscr{C}$ be the C^*-algebra made of functions on $\mathbb{R}^2$ having* a cartesian anisotropy, i.e. *$V \in \mathscr{C}$ if and only if there exists $V_1^\pm$, $V_2^\pm$ in $BC_u(\mathbb{R})$ such that*

$$\lim_{x\to\pm\infty} \sup_{y\in\mathbb{R}} \big|V(x,y) - V_2^\pm(y)\big| = 0 \quad \text{and} \quad \lim_{y\to\pm\infty} \sup_{x\in\mathbb{R}} \big|V(x,y) - V_1^\pm(x)\big| = 0.$$

In this case, the compact space Ω *is rather easy to describe, namely*

$$\Omega = [-\infty, \infty] \times [-\infty, \infty],$$

and if one sets $H_j^{\pm} = h(D) + V_j^{\pm}(X)$, *then* (4.5.5) *reads:*

$$\sigma_{ess}(H) = \sigma(H_1^+) \cup \sigma(H_1^-) \cup \sigma(H_2^+) \cup \sigma(H_2^-).$$

Exercise 4.5.6. *Consider the cartesian anisotropy in an arbitrary dimension, as introduced in Section 3 of [Ric05].*

In the previous example, the space Ω was easily understandable. However, even if Ω is not so explicit, computations can be performed based on our understanding of quasi-orbits. We present a final example in this direction.

Example 4.5.7 (Vanishing oscillations)**.** *Let us consider the* C^**-algebra* $\mathscr{C} = VO(\mathbb{R}^d)$ *of functions with vanishing oscillations,* i.e. $V \in VO(\mathbb{R}^d)$ *if and only if* $V \in C_b(\mathbb{R}^d)$ *and for any* $x \in \mathbb{R}^d$, *the function* $V(\cdot - x) - V(\cdot)$ *belongs to* $C_0(\mathbb{R}^d)$. *Clearly,* $VO(\mathbb{R}^d)$ *is a unital* $\mathbb{R}^d$*-invariant* C^**-subalgebra of* $BC_u(\mathbb{R}^d)$, *and contains* $C_0(\mathbb{R}^d)$. *Therefore,* Ω *is a compactification of* $\mathbb{R}^d$, *and each point of* $\partial\Omega$ *is an orbit in itself. Indeed, if* $\tau \in \partial\Omega$, *then* $\tau(\varphi) = 0$ *for any* $\varphi \in C_0(\mathbb{R}^d)$, *and then by* (4.3.1) *one has for any* $x \in \mathbb{R}^d$ *and* $\varphi \in VO(\mathbb{R}^d)$*:*

$$[\theta_x(\tau)](\varphi) = \tau\big(\varphi(\cdot - x)\big) = \tau\big(\varphi(\cdot - x) - \varphi(\cdot)\big) + \tau\big(\varphi(\cdot)\big) = \tau(\varphi).$$

Thus, the only covering of $\partial\Omega$ *is obtained by* $\{\tau_i\}_{i\in\partial\Omega}$, *the asymptotic potentials are just constants, and in this case* (4.5.5) *reads:*

$$\sigma_{ess}(H) = \overline{\cup_{i\in\partial\Omega}\,\sigma\big(h(D) + V|_{\tau_i}\big)} = \big[\min h + \min V(\mathbb{R}^d)_{asy}, \infty\big)$$

where $V(\mathbb{R}^d)_{asy} := \cap_K \overline{V(\mathbb{R}^d \setminus K)}$ *and* K *are arbitrary compact neighbourhoods of* 0 *in* $\mathbb{R}^d$.