DOUBLE CENTRALIZERS AND EXTENSIONS OF C^*-ALGEBRAS

BY
ROBERT C. BUSBY

1. **Introduction.** If A, B, and C are associative algebras over a field such that A is a two-sided ideal in B and B/A is isomorphic with C, then we may speak of B as an extension of A by C. It is of interest to be able to find all such extensions (with certain ones identified by a suitable equivalence relation) given A and C. This problem was investigated and essentially solved by G. Hochschild in one of his early papers on cohomology of algebras (see [5]). In this paper we investigate the analogous theory for C^*-algebras. An effective solution to this problem should prove useful in the classification of such algebras.

One of the tools used by Hochschild in his investigations was what he called the algebra of multiplications, $M(A)$, of an associative algebra A. If A is C^*-algebra, then $M(A)$ is exactly the double centralizer algebra of A, a concept developed for semigroups and algebras by B. E. Johnson (see [6]). This concept seems to be very useful in analysis, and is related to the left centralizer concept which J. G. Wendel used to investigate group algebras. The commutative version of this idea was first introduced by S. Helgason under the name of the algebra of multipliers (see [4]), and was recently developed by J. Wang (see [9]) and others. The connection with the extension problem and the work of Hochschild has apparently not been made by those using the centralizer concept. The definitions of double centralizer given in this paper are Johnson's. The proof that $M(A)$ is an involutive algebra is also due to him, as is most of the proof of Propositions 2.5, 3.1. We show that $M(A)$ is a C^*-algebra if A is, and the abstract proof presented is new, however the fact can be deduced by using a result of G. A. Reid (see [7, Proposition 3, p. 1021]). We then use what we call the strict topology to investigate $M(A)$ for certain particular A. This topology is a generalization of a topology with the same name which was defined in the commutative case by R. C. Buck, and the resulting proof of 3.9 is new, although the result was proved by Johnson.

The main result of the paper is that the classes of extensions of a C^*-algebra A by a C^*-algebra C are in one to one correspondence with the $*$-homomorphisms from C to the quotient algebra $M(A)/A$. The author would like to thank Professor Peter Freyd for the suggestion of using pullback arguments in this section (§3). This eliminated several pages of long calculations. The remaining sections of this paper are devoted to the description of extensions and their primitive ideal spaces by the use of the main result. This paper formed the basis for the author's doctoral

Received by the editors March 1, 1967.

dissertation at the University of Pennsylvania. I would like to take this opportunity to thank Professor Edward Effros for his encouragement during the writing of this dissertation and for many helpful conversations and suggestions. I would also like to thank the referee for suggesting many excellent improvements in the paper, particularly in the statement and proof of Theorem 3.9 and in the proofs of Theorems 3.3 and 6.3 and Proposition 6.2.

2. The double centralizer algebra of a C^*-algebra.

2.1. DEFINITION. Let A be a C^*-algebra. By a double centralizer on A, we mean a pair (T', T'') of functions from A to A such that $xT'(y)=T''(x)y$ for x, y in A.

2.2. NOTATION. Throughout this paper, A will always denote a C^*-algebra, and the set of all double centralizers of A will always be denoted by $M(A)$.

2.3. LEMMA. *Let the closed unit ball of A be denoted by $S(A)$, and let $x \in A$. Then $\|x\|=\sup_{y\in S(A)} \|yx\|$.*

Proof. See [2, p. 6, 1.3.5].

2.4. COROLLARY. *If x, y are in A, and if $zx=zy$ for all $z \in A$, then $x=y$.*

Proof. If $z(x-y)=0$ for all z in A, then by 2.3, $\|x-y\|=0$.

2.5. PROPOSITION. *Let $(T', T'') \in M(A)$. Then*

(i) *T' and T'' are continuous linear maps from A to A.*

(ii) *$T'(xy)=T'(x)y$ for all x, y in A.*

(iii) *$T''(xy)=xT''(y)$ for all x, y in A.*

Proof. We prove only the statements made about T'. The statements made about T'' are proved analogously.

(i) Let $x \in A$, $y \in A$, $z \in A$. Let $(x_i)_{i\in I}$ be a net in A, and let α and β be complex numbers. Then $zT'(\alpha x+\beta y)=T''(z)(\alpha x+\beta y)=\alpha T''(z)x+\beta T''(z)y=z(\alpha T'(x)+\beta T'(y))$ and since z is arbitrary, linearity follows from 2.4. If

$$\lim_{i\to\infty} \|x_i-x\| = \lim_{i\to\infty} \|T'(x_i)-y\| = 0,$$

then

$$\begin{aligned}\|zT'(x)-zy\| &\leqq \|zT'(x)-zT'(x_i)\|+\|zT'(x_i)-zy\| \\ &\leqq \|T''(z)\|\,\|x-x_i\|+\|z\|\,\|T'(x_i)-y\|.\end{aligned}$$

Since the last term of this inequality tends to zero, we have

$$zT'(x) = zy \quad \text{for all } z \text{ in } A.$$

Thus $y=T'(x)$ and T' has a closed graph. By a well known theorem, T' is continuous.

(ii) Let $x \in A$, $y \in A$, $z \in A$. Then

$$zT'(xy) = T''(z)(xy) = (T''(z)x)y = (zT'(x))y = z(T'(x)y).$$

Therefore since z is arbitrary, $T'(xy)=T'(x)y$ and the proof is completed.

We now know that if $(T', T'') \in M(A)$, then T' and T'' are continuous and linear and may therefore be given the usual norm: $\|T'\| = \sup_{x \in S(A)} \|T'(x)\|$ and similarly for T''.

2.6. LEMMA. *If $(T', T'') \in M(A)$, then $\|T'\| = \|T''\|$.*

Proof. Let $x \in S(A)$. Then

$$\|T'(x)\| = \sup_{y \in S(A)} \|yT'(x)\| = \sup_{y \in S(A)} \|T''(y)x\| \leqq \sup_{y \in S(A)} \|T''(y)\| = \|T''\|.$$

Thus $\|T'\| \leqq \|T''\|$. The reverse inequality is proved analogously, and the lemma is complete.

2.7. DEFINITION. Let L be a continuous linear function from A to A. Let L^* be the function from A to A defined by $L^*(x) = (L(x^*))^*$. Now let $(T', T'') \in M(A)$. Then we have the following result.

2.8. LEMMA. *If $(T', T'')^*$ denotes the pair (T''^*, T'^*), then $(T', T'')^* M(A)$.*

Proof. Straightforward.

2.9. LEMMA. *Let $(T', T'') \in M(A)$ and $(S', S'') \in M(A)$. Then $(T'S', S''T'')$ is in $M(A)$.*

Proof. Straightforward.

2.10. DEFINITION. Let $(T', T'') \in M(A)$, $(S', S'') \in M(A)$, α a complex number. We define $*$-algebra and norm structures on $M(A)$ as follows

(i) $(T', T'') + (S', S'') = (T' + S', T'' + S'')$,
(ii) $\alpha(T', T'') = (\alpha T', \alpha T'')$,
(iii) $(T', T'')(S', S'') = (T'S', S''T'')$,
(iv) $(T', T'')^* = (T''^*, T'^*)$,
(v) $\|(T', T'')\| = \|T'\| = \|T''\|$.

2.11. THEOREM. *$M(A)$ provided with the operations and norm defined above is a C^*-algebra with identity.*

Proof. We have shown that (iii) and (iv) are valid definitions, and this is clear for (i) and (ii). It is trivial that (iv) gives an involution and (v) defines a norm with $\|TS\| \leqq \|T\| \, \|S\|$ for every $T \in M(A)$, $S \in M(A)$. We therefore clearly have an associative $*$-algebra with identity (I, I) where $I(x) = x$ for $x \in A$. We need only show that $M(A)$ is complete in the norm and that $\|T^*T\| = \|T\|^2$ for T in $M(A)$.

To show completeness, let $((T'_n, T''_n))_{1 \leqq n \leqq \infty}$ be a Cauchy sequence in $M(A)$. Then $(T'_n)_{1 \leqq n \leqq \infty}$ and $(T''_n)_{1 \leqq n \leqq \infty}$ are Cauchy sequences in $L(A)$ (the C^*-algebra of all bounded linear operators on A), and so converge uniformly to T'_∞ and T''_∞ in $L(A)$. If $x \in A$, $y \in A$, then

$$xT'_\infty(y) = \lim_n xT'_n(y) = \lim_n T''_n(x)y = T''_\infty(x)y$$

and so $(T'_\infty, T''_\infty) \in M(A)$. Clearly (T'_n, T''_n) tends to (T'_∞, T''_∞) in $M(A)$, and therefore

$M(A)$ is complete. In view of the definitions of norm and involution, we must show that $\|T''^*T'\| = \|T'\|^2$ for all $(T', T'') \in M(A)$ in order to show that $M(A)$ is a C^*-algebra. By Dixmier, [2, p. 6] it is enough to show that $\|T''^*T'\| \leqq \|T'\|^2$.

We first remark that $\|T''^*\| = \|T''\|$. In fact if $x \in S(A)$, then $\|T''^*(x)\| = \|T''(x^*)^*\| = \|T''(x^*)\|$. Since $S(A)$ is selfadjoint, the result follows. Then $\|T''^*T'\| \leqq \|T''\| \|T'\| = \|T'\|^2$, and the theorem is proved.

3. **Properties and examples of double centralizers.** If, as always, A is a C^*-algebra and $M(A)$ its double centralizer algebra, then we define a map $\mu_0 : A \to M(A)$ by the formula $\mu_0(x) = (L_x, R_x)$, where $L_x(y) = xy$ and $R_x(y) = yx$ for all $y \in A$.

3.1. PROPOSITION. (i) *μ_0 is an injective $*$-homomorphism and $\mu_0(A)$ is a closed, two-sided ideal in $M(A)$.*

(ii) *μ_0 is surjective if and only if A has identity.*

(iii) *If A is commutative, then so is $M(A)$.*

We need the following preliminary result.

3.2. LEMMA. *If K and L are C^*-algebras, and $\Phi: K \to L$ is a $*$-homomorphism, then Φ is norm reducing and $\Phi(K)$ is closed in L. If Φ is injective, then it is an isometry.*

Proof. See [2, §1.8, p. 18] for the proof of this lemma.

Proof of 3.1. (i) We see immediately from the definitions and the associative law in A, that μ_0 is a $*$-homomorphism. If $\mu_0(a) = 0$, then in particular $ax = 0$ for all x in A, and so $a = 0$. Thus by 3.2 μ_0 is an isometric isomorphism of A onto the closed subalgebra $\mu_0(A)$ of A. If $T = (T', T'') \in M(A)$, and y and x are in A, then $(T'L_x)y = T'(xy) = T'(x)y$ and also $(R_xT'')y = R_x(T''(y)) = T''(y)x = yT'(x)$. This means that

$$T\mu_0(x) = (T', T'')(L_x, R_x) = (T'L_x, R_xT'') = (L_{T'(x)}, R_{T'(x)}) = \mu_0(T'(x)).$$

Similarly $\mu_0(x)T = \mu_0(T''(x))$. Thus $\mu_0(A)$ is an ideal in $M(A)$.

In the future, we shall frequently identify A with $\mu_0(A)$. In this case we will write $T'(x)$ as Tx and $T''(x)$ as xT. The defining equation for a double centralizer will then appear as the associative law for multiplying elements of A and $M(A)$.

(ii) If μ_0 is onto, then it is an isomorphism between A and $M(A)$. Since $M(A)$ has an identity, so does A.

Now suppose that A has an identity which we will denote by 1. If $(T', T'') \in M(A)$ and $x \in A$, then $T'(x) = T'(1x) = T'(1)x$. Thus $T' = L_{T'(1)}$ and similarly $T'' = R_{T'(1)}$ which means that μ_0 is onto.

(iii) Let $T \in M(A)$, $S \in M(A)$ and suppose that A is commutative. Let x and y be in A. Then we have

$$\begin{aligned}((ST)x)y &= S(Tx)y = S(y(Tx)) = (Sy)(Tx) = (Tx)(Sy) \\ &= T(x(Sy)) = T(Sy)x = (TS)(yx) = (TS)(xy) = ((TS)x)y.\end{aligned}$$

Since y is arbitrary, we have $(TS)x=(ST)x$ for all x in A. Thus $TS=ST$ for any $T\in M(A)$, $S\in M(A)$, and $M(A)$ is commutative. This completes the proof of 3.1.

3.4. DEFINITION. Let M be a C^*-algebra and let $N\subset M$ be a closed, two-sided ideal. We define the strict topology of M with respect to N, denoted $S(M:N)$, to be the locally convex topology generated by the seminorms $(\lambda_y)_{y\in N}$ and $(\rho_y)_{y\in N}$, where $\lambda_y(x)=\|yx\|$ and $\rho_y(x)=\|xy\|$. It is immediate that $S(M:N)$ is a vector space topology in which multiplication is separately continuous and which induces the $S(N:N)$ topology on N.

3.5. PROPOSITION. *With notation as above, N is $S(M:N)$ dense in M.*

Proof. We first remark that any C^*-algebra has an approximate identity (see [2, §1.7, pp. 15, 16]). Let $(u_\sigma)_{\sigma\in\Sigma}$ be an approximate identity in the C^*-algebra N. Then by definition

$$\lim_{\sigma\to\infty}\|xu_\sigma-x\| = 0 \quad\text{and}\quad \lim_{\sigma\to\infty}\|u_\sigma x-x\| = 0 \quad\text{for } x\in N.$$

Now if $y\in M$, then $yu_\sigma\in N$ and $u_\sigma y\in N$ and so $\lambda_x(y-yu_\sigma)=\|xy-(xy)u_\sigma\|$ tends to 0 and $\rho_x(y-yu_\sigma)=\|yx-yu_\sigma x\|\leqq\|y-yu_\sigma\|\;\|x\|$ which also tends to 0. Hence yu_σ tends strictly to y and the proof is complete.

3.6. PROPOSITION. *$M(A)$ is $S(M(A):A)$ complete.*

Proof. Let $(T_i)_{i\in I}$ be an $S(M(A):A)$ Cauchy net in $M(A)$. Then if $x\in A$, $(T_i'(x))_{i\in I}=(T_ix)_{\cdot\in I}$ is Cauchy in A and so converges to $T_\infty'(x)$ in A. Similarly $(T_i''(x))_{i\in I}$ converges in A to $T_\infty''(x)$. These definitions of T_∞' and T_∞'' show that both are linear functions from A to A, and if $x, y\in A$ we have

$$xT'(y) = \lim_{i\to\infty} xT_i'(y) = \lim_{i\to\infty} T_i''(x)y = T_\infty''(x)y.$$

Thus $T_\infty=(T_\infty', T_\infty'')$ is in $M(A)$ and clearly $(T_i)_{i\in I}$ converges strictly to T_∞. Q.E.D.

We remark that $M(A)$ is the unique completion of A relative to the $S(A:A)$ topology on A. See [10, Chapter 3, p. 79].

3.7. PROPOSITION. *Suppose B is a C^*-algebra containing A as a two-sided ideal. Then*

(i) *There is a unique $*$-homomorphism $\mu\colon B\to M(A)$ with the property that $\mu(x)=\mu_0(x)$ for all $x\in A$.*

(ii) *Let $A^\circ=\{x\in B\mid xA=0\}$. Then* $\ker\mu=A^\circ$.

(iii) $\mu\colon(B, S(B:A))\to(M(A), S(M(A):A))$ *is a continuous vector space morphism which is open onto its image.*

(iv) *Suppose that $A^\circ=0$, and B is $S(B:A)$ complete. By* (iii), *and* (ii), *μ is then an embedding of B in $M(A)$. We claim that under this embedding, $B=M(A)$.*

Proof. (i) For any $y\in B$, define the maps L_y and R_y from A to A exactly as they were defined in §3. Clearly $(L_y, R_y)\in M(A)$. If we let $\mu(y)=(L_y, R_y)$, then the map

μ so defined is clearly a $*$-homomorphism, and $\mu|A=\mu_0$ by the definitions. If γ is any other map with the above properties and if $y \in B$, $x \in A$, then

$$(\gamma(y)-\mu(y))\mu_0(x) = \gamma(yx)-\mu(yx) = \mu_0(yx)-\mu_0(yx) = 0.$$

Thus $\gamma(y)=\mu(y)$ for all y and so $\gamma=\mu$.

(ii) $y \in \ker \mu$ if and only if $L_y=0$, which is equivalent with $y \in A^\circ$.

(iii) A base for the topology $\boldsymbol{S}(B:A)$ of B about the origin is given by the sets $V(x_1,\ldots,x_n;r)=\{y \in B \mid \|x_i y\|+\|yx_i\| \leqq r\}$ where the x_i are in A, and $r>0$ is a real number. A similar statement holds for $\boldsymbol{S}(\mu(B),\mu_0(A))$ topology of $\mu(B)$. Now $\|x_i y+yx_i\|=\|\mu_0(x_i y)\|+\|\mu_0(yx_i)\|=\|\mu_0(x)\mu(y)\|+\|\mu(y)\mu_0(x_i)\|$, so μ is continuous and open onto $\mu(B)$.

(iv) If B is complete in $\boldsymbol{S}(B:A)$, then it may be considered to be a complete subspace of $(M(A), \boldsymbol{S}(M(A):A))$.

Now $M(A)$ is Hausdorff and a complete subset of a Hausdorff space is closed, so B is a closed subset of $M(A)$. On the other hand $B\supset A$ and A is dense in $M(A)$, so B is dense in $M(A)$. Thus $B=M(A)$ and the proof is completed.

3.8. PROPOSITION. *Let A and A' be C^*-algebras and $\alpha: A \to A'$ be an onto $*$-homomorphism. According to* [6, §2, *Theorem* 4, *p.* 302], *there is a unique map $\bar{\alpha}: M(A) \to M(A')$ which extends the map α. It is defined by $\bar{\alpha}(T)\alpha(x)=\alpha(Tx)$ and $\alpha(x)\bar{\alpha}(T)=\alpha(xT)$ for all $x \in A$, $T \in M(A)$. We claim that $\bar{\alpha}$ is continuous when the $M(A)$ is given the $\boldsymbol{S}(M(A):A)$ topology and $M(A')$ is given the $\boldsymbol{S}(M(A'):A')$ topology.*

Proof. Let $(T_i)_{i\in I}$ be a net in $M(A)$ and suppose that this net converges strictly to T_∞. Then if $x' \in A'$ and $x'=\alpha(x)$, we have $\|\bar{\alpha}(T_i)x'-\bar{\alpha}(T_\infty)x'\|=\|\alpha(T_i x)-\alpha(T_\infty x)\| \leqq \|T_i x-T_\infty x\|$ and the latter expression tends to zero. Similarly $\|x'\bar{\alpha}(T_i)-x'\bar{\alpha}(T_\infty)\|$ tends to zero and so by definition, $\bar{\alpha}(T_i)$ converges strictly to $\bar{\alpha}(T_\infty)$ in $M(A')$ and $\bar{\alpha}$ is continuous.

We use the strict topology to compute $M(A)$ for certain specific C^*-algebras A. Let H be a Hilbert space, $L(H)$ the C^*-algebra of all bounded linear operators on H, and $A=CC(H)$ the two-sided ideal in $L(H)$ consisting of all compact operators on H. The fact that $M(A)=L(H)$ in this case follows from the theorem below.

3.9. THEOREM. *If H is a Hilbert space and A is a C^*-subalgebra of $L(H)$, such that the space AH is closed in H, define $B=\{y \in L(H) \mid yA+Ay \subset A\}$ (the idealizer of A in $L(H)$). Then*

(i) *B is $\boldsymbol{S}(B:A)$ complete and*

(ii) *if* Cl $(AH)=H$, *then B is canonically isomorphic with $M(A)$.*

Proof. (i). Let (x_i) be an $\boldsymbol{S}(B:A)$ Cauchy net in B. Then if $L_x(a)=xa$ for $x \in B$, $a \in A$, we have L_{x_i} converges to an element $T \in L(A)$ by the Banach-Steinhaus theorem, where $T(a)=\lim_i x_i a$ for all $a \in A$. Now let $\xi \in AH$, say $\xi=a\eta$, $a \in A$, $\eta \in H$. Then $\lim_i x_i\xi$ exists and defines a linear operator S on AH by: $S\xi=\lim_i x_i\xi$

$=\lim_i (x_i a)\eta = T(a)\eta$. Since $\|S\xi\| \leqq \|T\| \, \|\xi\|$, we have $\|S\| \leqq \|T\|$ and $S \in L(AH)$, and so we can extend S to an operator $\bar{S}$ on $\mathrm{Cl}\,(AH)$. Define $b \in L(H)$ to be $\bar{S}$ on $\mathrm{Cl}\,(AH)$ and to be zero on $\mathrm{Cl}\,(AH)^{\perp}$. If $a \in A$ and $\eta \in H$, we have $(ba)\eta = b(a\eta) = S(a\eta) = T(a)\eta$. Therefore we have $ba = T(a) = \lim_i x_i a \in A$. Since (x_i) is $\boldsymbol{S}(B:A)$ Cauchy we know that ax_i converges to an element of A for every $a \in A$, so if $a, a' \in A$, we have $(ab)a' = a(ba') = \lim_i a(x_i a') = \lim_i (ax_i)a' = (\lim_i ax_i)a'$. Since this holds for all $a' \in A$, we know that $ab = \lim_i ax_i \in A$, and so $ab \in A$. Thus $b \in \boldsymbol{B}$, and (x_i) obviously converges strictly to b, so B is $\boldsymbol{S}(B:A)$ complete.

(ii) If $\mathrm{Cl}\,(AH) = H$, then since $A^{\circ}H$ is perpendicular to $\mathrm{Cl}\,(AH)$, $A^{\circ}H = \{0\}$. This means that $A^{\circ} = 0$ and so by (i) and Proposition 3.7.iv. B is canonically isomorphic with $M(A)$, which completes the theorem.

A theorem essentially the same as the above has been proved by G. A. Reid using a very similar argument, see [7, Proposition 3, p. 1021].

We now briefly summarize the theory of continuous fields of C^*-algebras in order to state our next few results. As general references, we cite [2, Chapter 10] and [3].

3.10. Definition. Let S be a locally compact, Hausdorff space, and for all $s \in S$ let $F(s)$ be a C^*-algebra. A continuous S-field of C^*-algebras is a pair $(\boldsymbol{F}, \theta)$ where $\boldsymbol{F} = (F(s))_{s \in S}$ is a family of C^*-algebras and θ is a collection of maps $s \to x(s)$ on S, with $x(s) \in F(s)$ for all $s \in S$ and such that

(1) The map $s \to \|x(s)\|$ is continuous on S for all $x \in \theta$.

(2) θ is closed under pointwise addition, scalar multiplication, and involution.

(3) For all $s \in S$, $\{x(s) \mid x \in \theta\}$ is dense in $F(s)$.

(4) Suppose a map $s \to y(s)$ with $y(s) \in F(s)$ has the following property: For all $s_0 \in S$ and $\varepsilon > 0$, there is an $x \in \theta$ and a neighborhood U of s_0 such that $\|x(s) - y(s)\| < \varepsilon$ for all $s \in U$. Then $y \in \theta$.

We recall that a C^*-algebra is said to be C.C.R. if its image under every irreducible representation consists of all the compact operators on some Hilbert space H. The set of equivalence classes of irreducible representations, suitably topologized, is often called the spectrum of the algebra. If the spectrum of a C^*-algebra A satisfies the T_0 separation axiom for topological spaces, then it may be thought of as the set of primitive ideals with the Jacobson topology (see [2, §3.1, pp. 59, 60]), and we will think of it this way in the future.

3.11. Proposition. *Let A be a C.C.R. algebra whose spectrum*, $\mathrm{sp}\,(A)$, *is Hausdorff. Each $s \in \mathrm{sp}\,(A)$ is a primitive ideal and the image of the corresponding representation on the Hilbert space H_s is the algebra A/s. Then we claim that there is a continuous* $\mathrm{sp}\,(A)$*-field of C^*-algebras, (F, θ) such that for any $s \in \mathrm{sp}\,(A)$, $F(s) = A/s$, and A is isomorphic with the set of all $x \in \theta$ having the property that the function $s \to \|x(s)\|$ vanishes at infinity. This set does form a C^*-algebra denoted A_F if we define the algebraic operations pointwise, and the norm by*:

$$\|x\| = \sup_{s \in \mathrm{sp}(A)} \|x(s)\|.$$

Proof. For the proof of this proposition, see [2, §10.5, pp. 201, 202].

We can also proceed in the opposite direction and start with an S-continuous field $(F:\theta)$ of C^*-algebras. We then construct as above, the C^*-algebra A_F. If $(F:\theta)$ has the property that for all $s \in S$, $F(s)$ consists of the compact operators on some Hilbert space H_s, then S is the spectrum of A_F. If we now let $A = A_F$ in Proposition 3.11, then the construction of that proposition will give back the field $(F:\theta)$ with which we started. We therefore can, and in the future shall, identify the C.C.R. algebras having T_2 spectrum, with the algebras A_F for all S-fields for which each $F(s)$ is the algebra of compact operators on H_s. Such S-fields will be called C.C.R. fields on S.

Using the above notation, let $(F:\theta)$ be a C.C.R. field on S, and let $A = A_F$. For each $s \in S$, $L(H_s)$ is the algebra of all bounded linear operators on the Hilbert space H_s. We define $M'(A)$ to be the set of all functions $s \to y(s)$ on S such that

(1) For all $s \in S$, $y(s) \in L(H_s)$.

(2) For all $x \in A$, we have $xy \in A$, and $yx \in A$.

3.12. Theorem. (i) *For any $y \in M'(A)$, the function $s \to \|y(s)\|$ is bounded on S. Furthermore, $M'(A)$ becomes a C^*-algebra if we define operations pointwise and use the norm*: $\|y\| = \sup_{s\in S} \|y(s)\|$.

(ii) *$M'(A)$ is isomorphic with $M(A)$.*

We need a preliminary lemma.

3.13. Lemma. *Let S be a locally compact, Hausdorff space, and let $(F:\theta)$ be an S-continuous field of C^*-algebras. Suppose $s_0 \in S$, and $x_0 \in F(s_0)$. Then there is an $x \in A_F$ such that $\|x\| = \|x_0\|$ and $x(s_0) = x_0$.*

Proof. We will not give the proof of this lemma here. A proof may be worked out using [2, §10.1, Propositions 10.1 and 10.1.9] and standard techniques of Urysohn functions on locally compact, Hausdorff spaces.

Proof of Theorem 3.12. (i) Let $s \in S$, and $\sigma \in H_s$. P_σ will denote projection of H_s onto the one dimensional subspace spanned by σ. Then $\|P_\sigma\| = 1$ and by 3.13 there is an $x \in A_F$ with $\|x\| = 1$ and $x(s) = P_\sigma$. Now $\|y(s)\sigma\| = \|y(s)P_\sigma\sigma\| \leqq \|y(s)P_\sigma\| \, \|\sigma\| = \|yx(s)\| \, \|\sigma\|$. Thus $\|y(s)\| \leqq \|yx(s)\| \leqq \|yx\|_\infty$ (the supremum norm in A_F). To show that $\|y(s)\|$ is a bounded function on S, it is therefore enough to prove that: $\sup_{x\in S(A)} \|yx\| < M$. This follows from condition (2) on $M'(A)$ and the continuity argument of 2.5.(i).

Now it is clear from the definitions that $M'(A)$ is a vector space. If y_1 and y_2 are in $M'(A)$, and x is in A, then $(y_1y_2)x = y_1(y_2x) \in A$. Similarly $x(y_1y_2) \in A$. Therefore $y_1y_2 \in M'(A)$. Also we have $(y_1^*x) = (x^*y_1)^* \in A$, and similarly $xy_1^* \in A$. This means that $y_1^* \in M'(A)$ and so $M'(A)$ is a $*$-algebra. For each $s \in S$, we have $\|y(s)^*y(s)\| = \|y(s)\|^2$ which means that $\|y^*y\| = \|y\|^2$ for all $y \in M'(A)$. To complete the proof that $M'(A)$ is a C^*-algebra, it is only necessary to show that it is complete in the supremum norm. Let (y_n) be a norm Cauchy sequence in $M'(A)$, and let $x \in A$. Then (xy_n) is a norm Cauchy sequence in A. If $s \in S$, then $(y_n(s))$ is Cauchy in $L(H_s)$ and therefore converges to some $y_\infty(s)$. Denote the function $s \to y_\infty(s)$ by y_∞. Clearly the sequence (xy_n) converges pointwise to xy_∞, and since (xy_n) is norm Cauchy, it must converge in norm to xy_∞. By completeness of A, $xy_\infty \in A$ for all $x \in A$, and similarly $y_\infty x \in A$ for all $x \in A$, so $y_\infty \in M'(A)$. Since (y_n) is norm Cauchy and converges pointwise to y_∞, it converges to y_∞ in norm, and the proof is complete.

(ii) We wish, as before, to use Proposition 3.7 to show that $M'(A)$ may be identified with $M(A)$. We know that $F(s)=CC(H_s)$, for any $s \in S$, and so we may show as in Theorem 3.9 that if $y_0 \in L(H_s)$ and $y_0x_0=0$ for all $x_0 \in F(s)$, then $y_0=0$. We also know, by Lemma 3.13, that for any $x_0 \in F(s)$ there is an $x \in A$ with $x(s)=x_0$. Combining, we see that if $y \in M'(A)$ and $yx=0$ for all $x \in A$, then $y(s)x_0=0$ for all $x_0 \in F(s)$ and $s \in S$. This means that $y(s)=0$ for all $s \in S$, and so $y=0$. The first requirement for the use of 3.7 is therefore satisfied, and it remains only to show that $M'(A)$ is $\boldsymbol{S}(M'(A):A)$ complete. Let $(y_\iota)_{\iota \in I}$ be an $\boldsymbol{S}(M'(A):A)$ Cauchy net in $M'(A)$. We then have that for all $s \in S$, $(y_\iota(s))_{\iota \in I}$ is $\boldsymbol{S}(L(H_s) : F(s))$ Cauchy and so converges in this topology to $y_\infty(s)$. Again we denote the function $s \to y_\infty(s)$ by y_∞. For all $x \in A$, $(y_\iota x)$ is a norm Cauchy net converging pointwise to $y_\infty x$, and so $y_\infty x \in A$. Similarly $xy_\infty \in A$, and so $y_\infty \in M'(A)$, and clearly $(y_\iota)_{\iota \in I}$ converges strictly to y_∞. This completes the proof of Theorem 3.12.

3.14. DEFINITION. We say that a C^*-algebra A is n-homogeneous if every irreducible representation has as image the set M_n, of all $n \times n$ complex valued matrices.

It is proved in [2, §36, pp. 74, 75] that every n-homogeneous C^*-algebra is automatically C.C.R. and has a Hausdorff spectrum. If A is n-homogeneous with spectrum S, we know by previously cited results that there is an s-continuous field $(\boldsymbol{F}, \theta)$ of C^*-algebras with $A=A_{\boldsymbol{F}}$. It is known (see [3, §3.2, p. 249]) that if we let $\boldsymbol{F}=\bigcup_{s \in S} F(s)$, then with the obvious projection of $\boldsymbol{F}$ onto S, $\boldsymbol{F}$ can be made into a fibre bundle over S with fibre M_n and group G_n of all automorphisms of M_n of the form $a \to u^{-1}au$, u a unitary matrix. This can be done in such a way that all $x \in \theta$ are continuous sections of the bundle, and $A=S_\infty(S, \boldsymbol{F})$, the set of all continuous sections which vanish at infinity in norm. Let $S_b(S, \boldsymbol{F})$ denote the set of all continuous sections of F over S which are bounded in norm. Then we have the following result.

3.15. THEOREM. *Let A be an n-homogeneous C^*-algebra with spectrum S. Let $\boldsymbol{F}$ be the bundle over S which we associated with A above. Then we know that $A=S_\infty(S, \boldsymbol{F})$. We claim that $M(A)=S_b(S, \boldsymbol{F})$.*

Proof. We know by Theorem 3.12 that $M(A)$ consists of the set of all sections $s \to y(s)$ over S, with the property that $xy \in A$ and $yx \in A$, for all $x \in A$, and we also know that such y are bounded in norm. It is certainly clear that every y in $S_b(S, \boldsymbol{F})$ has this property. To complete the theorem, we need only show that every function $s \to y(s)$ with the above property is continuous. Let $s \to y(s)$ be such a section, and let $s_0 \in S$. Since a fibre bundle is locally trivial, we may find a compact neighborhood

V of s_0 such that A restricted to V is trivial, i.e. A restricted to V consists of all norm continuous functions from V to M_n. In particular, there is an element $x \in A$ such that if we denote the identity matrix by I, we have $x(s)=I$ for all $s \in V$. Now since S is locally compact and Hausdorff, there is a compact neighborhood U of s_0 contained in V and a continuous function $f: S \to [0, 1]$ such that $f(s)=1$ for $s \in U$ and f is identically zero outside of the interior of V. Then $fx \in A$ (see [2, Chapter 10]), and so $fyx \in A$. Since $\|fyx(s)\|$ is a continuous function, and $fyx=y$ on U, we see that the function $s \to \|y(s)\|$ is continuous at s_0. Since s_0 was arbitrary, the proof is complete.

4. **Extensions of C^*-algebras.** If A and C are C^*-algebras, then by an extension. of A by C we mean a C^*-algebra B such that A is a closed, two-sided ideal in B, and $B/A=C$. An example of such an extension is the C^*-product $A \times C$. This is the cartesian product of A and C with the algebraic operations defined pointwise, and the supremum norm.

In order to facilitate the discussion, we give a more formal definition of extension.

4.1. Definition. Let A and C be C^*-algebras. An extension of A by C is a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of C^*-algebras. That is, f and g are $*$-homomorphisms, f is one to one, g is onto, and the image of f is the kernel of g. This implies that the image, $f(A)$, of f is a closed two-sided ideal in B, and $B/f(A) \simeq C$. We will usually identify A with $f(A)$, but the f will be understood since there are many such embeddings.

Given two extensions $E_1: 0 \to A \xrightarrow{f_1} B_1 \xrightarrow{g_1} C \to 0$ and $E_2: 0 \to A \xrightarrow{f_2} B_2 \xrightarrow{g_2} C \to 0$ of A by C, we say that they are equivalent if there is a $*$-isomorphism $\theta: B_1 \to B_2$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle I_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \longrightarrow & 0 \end{array}$$

The vertical maps on the two ends represent the identity isomorphisms. This relationship is an equivalence relation and the set of equivalence classes is usually denoted by Ext (C, A).

4.2. Proposition. *Let* $E: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an extension of* A *by* C, *where* A *and* C *are* C^*-*algebras, and let* $\gamma: C' \to C$ *be a* $*$-*homomorphism of a* C^*-*algebra* C' *into* C. *Then there is a* C^*-*algebra* B', *and* $*$-*homomorphisms* $f': A \to B'$, $g': B' \to C'$, *and* $\theta: B' \to B$ *such that*

(i) *the sequence* $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ *is an extension of* A *by* C'.

(ii) *The following diagram is commutative*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

Furthermore, the extension $E': 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ *is unique up to equivalence.*

Proof. Existence. Let $B' = \{(b, c') \in B \times C' \mid g(b) = \gamma(c')\}$. B' is customarily called the pullback of the maps g and γ. Since g and γ are $*$-homomorphisms, and therefore continuous, we see that B' is a C^*-subalgebra of $B \times C'$. Define the maps f', g', and θ by the following formulas: $f'(a) = (f(a), 0)$, $g'((b, c')) = c'$, $\theta((b, c')) = b$. It is easy to check that these definitions are valid (the only question is whether f' maps into B'), that they are $*$-homomorphisms, and that the sequence $E': 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ is exact. In fact to prove the latter, we first note that f' and g' are respectively injective and surjective because of the definitions and the corresponding statements for f and g. It is immediate that $g'f' = 0$. To show that $\operatorname{Im}(f') = \ker(g')$, we notice that if $g'((b, c')) = 0$ then $g(b) = \gamma(c') = 0$. By the exactness of E at B we must have $b = f(a)$ for some $a \in A$ and so $(b, c') = (f(a), 0) = f'(a)$, and exactness is proved. The commutativity asserted in (ii) is clear by the construction of the maps in question.

Uniqueness. Let $E'': 0 \to A \xrightarrow{f''} B'' \xrightarrow{g''} C' \to 0$ be another exact sequence, and suppose that there is a $*$-homomorphism $\theta'': B'' \to B$ such that the diagram

$$\begin{array}{ccccccccc} E'': 0 & \longrightarrow & A & \xrightarrow{f''} & B'' & \xrightarrow{g''} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_A} & & \downarrow{\scriptstyle \theta''} & & \downarrow{\scriptstyle I_C} & & \\ E: 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

commutes. If we define $\alpha: B'' \to B'$ by $\alpha(b'') = (\theta'(b''), g''(b''))$, then α is a $*$-isomorphism and the diagram

$$\begin{array}{ccccccccc} E'': 0 & \longrightarrow & A & \xrightarrow{f''} & B'' & \xrightarrow{g''} & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle I_A} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle I_C} & & \\ E': 0 & \longrightarrow & A & \xrightarrow{f} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

commutes. Thus E' is unique up to equivalence and the proposition is proved.

We shall denote by E_γ the extension E' constructed above from the extension E and the $*$-homomorphism γ, and its equivalence class is denoted by $[E_\gamma]$. We now come to the main result of the paper. Let A be a C^*-algebra. We know that A is a two-sided ideal in $M(A)$, and we have denoted the natural injection of A into $M(A)$ by μ_0. Let us denote the quotient algebra $M(A)/A$ by $O(A)$, the natural projection of $M(A)$ onto $O(A)$ by π, and the corresponding exact sequence $0 \to A \xrightarrow{\mu_0} M(A) \xrightarrow{\pi} O(A) \to 0$ by E^0. We refer to $O(A)$ as the algebra of outer centralizers of A. If P and Q are C^*-algebras, we let the set of all $*$-homomorphisms from P to Q be denoted by $\operatorname{Hom}(P, Q)$.

4.3. THEOREM. *The mapping* $\gamma - [E_\gamma^0]$ *from* $\operatorname{Hom}(C, O(A))$ *to* $\operatorname{Ext}(C, A)$ *is a bijection.*

Proof. We first show that the mapping is injective. Suppose that $\gamma, \gamma' : C \to O(A)$ are $*$-homomorphisms with E_γ^0 and $E_{\gamma'}^0$ equivalent. Then it is immediate from the definitions and the explicit construction of the pullbacks that $\gamma = \gamma'$. We now show that the mapping is surjective. Let $E: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be in Ext (C, A). By Proposition 3.7 there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow I_A & & \downarrow \mu & & & & \\ 0 & \longrightarrow & A & \xrightarrow{\mu_0} & M(A) & \xrightarrow{\pi} & O(A) & \longrightarrow & 0 \end{array}$$

which can be filled in by a unique morphism $\gamma: C \to O(A)$ under preservation of continuity, since $\pi\mu$ vanishes on $f(A) = \ker g$. By the uniqueness assertion of 4.2. E and E_γ^0 are equivalent. This completes the proof of the theorem.

5. **Properties of extensions.** Suppose that the following diagram is commutative:

$$\begin{array}{ccccccccc} E_1: 0 & \longrightarrow & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \longrightarrow & 0 \\ & & \downarrow \omega & & \downarrow \theta & & \downarrow \sigma & & \\ E_2: 0 & \longrightarrow & A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \longrightarrow & 0 \end{array}$$

Furthermore, E_1 and E_2 are assumed to be exact sequences of C^*-algebras and ω, θ, and σ are $*$-isomorphisms. We will then say that E_1 and E_2 are weakly equivalent, taking care to distinguish this relation from the earlier one which we called equivalence. In less technical terms, E_1 and E_2 are weakly equivalent if there exists a $*$-isomorphism θ from B_1 to B_2 which induces an isomorphism of A onto itself, and the corresponding isomorphism of C to C. Since extensions are involved in the problem of classification of C^*-algebras up to isomorphism, the weak equivalence seems to be a natural concept. In addition, we will show that in general the set of weak equivalence classes is much smaller and easier to compute than the set of equivalence classes.

Let $\gamma_1, \gamma_2 \in$ Hom $(C, O(A))$. We know (see Proposition 3.8) that if $\omega: A \to A$ is a $*$-isomorphism, there is a unique extension $\bar{\omega}: M(A) \to M(A)$ of ω. Since $\bar{\omega}$ takes A onto A, it induces a $*$-homomorphism $\hat{\omega}: O(A) \to O(A)$ in the usual way. We shall say that γ_1 and γ_2 are weakly equivalent if there exist $*$-homomorphisms $\omega: A \to A$, and $\sigma: C \to C$ such that $\gamma_2 = \hat{\omega}\gamma_1\sigma^{-1}$.

5.1. Proposition. *Let E_1 and E_2 be extensions of A by C with equivalence classes $[E_1]$ and $[E_2]$. Let γ_1 and γ_2 be the maps in* Hom $(C, O(A))$ *which correspond to $[E_1]$ and $[E_2]$ under the correspondence of Theorem* 4.3, *then* (i) *E_1 is weakly equivalent with E_2 if and only if every extension in $[E_1]$ is weakly equivalent with every extension in $[E_2]$. In this case we say that $[E_1]$ and $[E_2]$ are weakly equivalent.* (ii) *$[E_1]$ is weakly equivalent with $[E_2]$ if and only if γ_1 is weakly equivalent with γ_2.*

Proof. (i) is clear from the definitions of equivalence and weak equivalence.

We prove (ii). Suppose that $[E_1]$ and $[E_2]$ are weakly equivalent. Then the following diagram is commutative:

$$\begin{array}{ccccccccc} E_1: 0 & \longrightarrow & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \longrightarrow & 0 \\ & & \omega\downarrow & & \theta\downarrow & & \downarrow\sigma & & \\ E_2: 0 & \longrightarrow & A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \longrightarrow & 0 \end{array}$$

Since the map $\bar{\omega}: M(A) \to M(A)$ which extends ω is unique (see Proposition 3.8), and since the above diagram is commutative, we see if we consider B_1 and B_2 as subalgebras of $M(A)$, then θ is $\bar{\omega}$ restricted to B_1. More formally, if μ_1 and μ_2 are the $*$-homomorphisms of B_1 and B_2 respectively into $M(A)$, guaranteed to exist by Proposition 3.7, then we are saying that $\bar{\omega}\mu_1 = \mu_2\theta$. We also remark that by the definition of the induced map $\hat{\omega}: O(A) \to O(A)$, we must have $\hat{\omega}\pi = \pi\bar{\omega}$.

Now let $u_1: C \to B_1$ be a linear map with $g_1u_1 = I_C$. If we denote γ_{E_1} by γ_1, γ_{E_2} by γ_2, and recall the construction of these maps, we see that we may let $\gamma_1 = \pi\mu_1u_1$. Let $u_2: C \to B_2$ be the linear map defined by the equation: $u_2 = \theta u_1\sigma^{-1}$. Then $g_2u_2 = g_2\theta u_1\sigma^{-1} = \sigma g_1u_1\sigma^{-1} = \sigma\sigma^{-1} = I_C$, and so we may define γ_2 by the equation $\gamma_2 = \pi\mu_2u_2$. We then have

$$\begin{aligned} \gamma_2 &= \pi\mu_2u_2 = \pi\mu_2\theta u_1\sigma^{-1} = \pi\bar{\omega}\mu_1u_1\sigma^{-1} \\ &= \hat{\omega}\pi\mu_1u_1\sigma^{-1} = \hat{\omega}\gamma_1\sigma^{-1}. \end{aligned}$$

Thus γ_1 is weakly equivalent with γ_2.

For the converse, suppose there are $*$-homomorphisms $\omega: A \to A$, and $\sigma: C \to C$ with $\gamma_2 = \hat{\omega}\gamma_1\sigma^{-1}$.

Let $E_i: 0 \to A \xrightarrow{f_i} B_1 \xrightarrow{g_i} C \to 0$ $(i=1, 2)$ be the extensions corresponding to γ_i. Recall that μ_0 is the natural injection of A into $M(A)$, and therefore since $\bar{\omega}$ extends ω, we have $\bar{\omega}\mu_0 = \mu_0\omega$.

Now define a map $\theta: B_1 \to B_2$ as follows

$$\theta(m, c) = (\bar{\omega}(m), \sigma(c)).$$

Since $(m, c) \in B_1$, we must have $\pi(m) = \gamma_1(c)$. Then $\pi\bar{\omega}(m) = \hat{\omega}\pi(m) = \hat{\omega}\gamma_1(c) = \gamma_2\sigma(c)$ (since $\gamma_2 = \hat{\omega}\gamma_1\sigma^{-1}$), which means that $\theta(m, c)$ does belong to B_2. Clearly θ is a $*$-homomorphism and we have $f_2\omega(a) = (\mu_0(\omega(a)), 0) = (\bar{\omega}(\mu_0(a)), 0) = \theta(\mu_0(a), 0) = \theta f_1(a)$. Similarly $g_2\theta(m, c) = g_2(\bar{\omega}(m), \sigma(c)) = \sigma(c) = \sigma g_1(m, c)$. From these computations it follows that the diagram

$$\begin{array}{ccccccccc} E_1: 0 & \longrightarrow & A & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C & \longrightarrow & 0 \\ & & \omega\downarrow & & \theta\downarrow & & \downarrow\sigma & & \\ E_2: 0 & \longrightarrow & A & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C & \longrightarrow & 0 \end{array}$$

is commutative. From this it follows that E_1 and E_2 are weakly equivalent, and completes the proof of the proposition.

We remark that for E_1 and E_2 to be weakly equivalent, it is not sufficient for B_1 and B_2 to be $*$-isomorphic, even if all the algebras are commutative. In fact we know that a commutative C^*-algebra is just a $C_\infty(X)$, that is the set of all complex valued functions vanishing at infinity on a locally compact, Hausdorff space X, where X is the maximal ideal space. Isomorphisms of commutative C^*-algebras correspond to homeomorphisms of the maximal ideal spaces, and closed two-sided ideals correspond to open sets in these spaces, while the complements of these open sets correspond to the appropriate quotient algebras. Considering all these things, we see that to find our counterexample, it is sufficient to find spaces X and Y and closed subsets F_1 and F_2 of X and Y respectively such that

(1) X and Y are homeomorphic.

(2) F_1 is homeomorphic with F_2, and the complements of F_1 and F_2 are homeomorphic to each other.

(3) No homeomorphism of X onto Y can take F_1 onto F_2.

One of the simplest such examples, suggested to the author by Carl Weisman, is shown in Figure 1. X and Y are both equal to the closed square about zero having

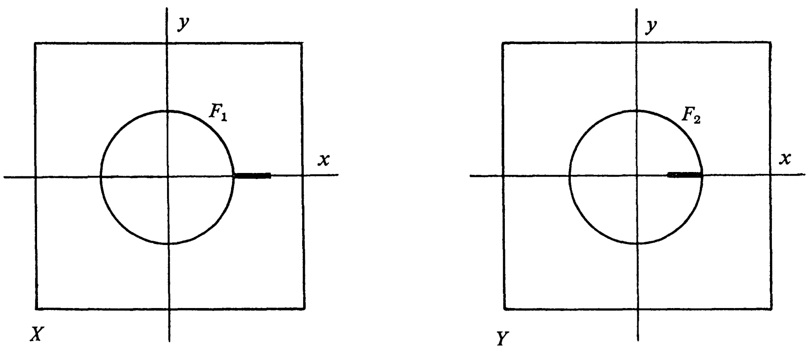

FIGURE 1

side of length four. F_1 is the unit circle with an outside spike, and F_2 is the unit circle with an inside spike.

5.2. DEFINITION. We say that an extension $E\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits if there is a $*$-homomorphism $u\colon C \to B$ with $gu = I_C$.

It is clear that if one extension has this property, then every extension equivalent to that one has the property. We then say that $[E]$ splits.

5.3. PROPOSITION. *The following three statements are equivalent*:

(i) *An extension E splits.*

(ii) *B is the direct sum, as a Banach space, of the ideal $f(A)$ and a C^*-subalgebra isomorphic with C.*

(iii) *If $\gamma\colon C \to O(A)$ corresponds to $[E]$, then there is a $*$-homomorphism $\rho\colon C \to M(A)$ with $\pi\rho = \gamma$. We say that such a map ρ is compatible with γ.*

Proof. (i)$\Rightarrow$(iii). Let $u: C \to B$ with $gu = I_C$. Since $\gamma = \pi\mu u$, we can let $\rho = \mu u$. This proves (iii).

(iii)$\Rightarrow$(i). Since splitting depends only on the class of an extension, we may assume that $E = E_\gamma$. Then a typical element of B is a pair $(m, c) \in M(A) \times C$, with $\pi(m) = \gamma(c)$. If ρ is compatible with γ, then we may define $u: C \to B$ by $u(c) = (\rho(c), c)$. Then $gu = I_C$ and u is clearly a $*$-homomorphism, which proves (i).

(ii)$\Rightarrow$(i). Suppose B is isomorphic as a Banach space to $f(A) \times D$ and $\beta: D \to C$ is a $*$-isomorphism, with inverse β^{-1}. An element b of B can be represented as $(f(a), d)$, and $g((f(a), d)) = \beta(d)$, therefore if we set $u = \beta^{-1}$, u will be a $*$-homomorphism of C into B with $gu = I_C$. Therefore (i).

(i)$\Rightarrow$(ii). It is a simple matter to show that the splitting map u gives the isomorphism of C with a C^*-subalgebra D and that B is isomorphic with $f(A) \times D$ as a Banach space. We omit the details.

As a final remark, we note that not all extensions split, even if the algebras are commutative. In fact it is known that if N represents the positive integers, $C_b(N)$ the C^*-algebra of all bounded sequences of complex numbers, and $C_\infty(N)$ is the ideal of all sequences converging to 0, then $C_\infty(N)$ has no closed complementary subspace in $C_b(N)$ and so the corresponding sequence cannot be split.

6. **Primitive ideals and extensions of C^*-algebras.** We now briefly review and amplify some remarks previously made about irreducible representations and primitive ideals.

Recall that if A is a C^*-algebra, we defined the spectrum of A to be the space of equivalence classes of irreducible representations of A on some Hilbert space. Denote this set by $\hat{A}$. Equivalent representations have the same kernel, and so we may speak of the kernel of $\pi \in \hat{A}$. We call any such kernel a primitive ideal and denote the set of all primitive ideals of A by Prim (A). Prim (A) has a natural T_0 topology (the Jacobson or hull kernel topology) in which the closure of a set $\{J_i\}_{i\in I}$ in Prim $(A) = \{J \in \text{Prim}(A) \mid J \supset \bigcap_{i\in I} J_i\}$. Prim (A) is not always a T_1 or T_2 (Hausdorff) space, and these conditions reflect structure properties of A. We remark that if A is commutative, then Prim (A) is the maximal ideal space of A with its usual topology, and is always T_2.

Suppose now that A and C are fixed C^*-algebras and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an extension of A by C. If $\pi \in \hat{C}$, then $\pi \circ g \in \hat{B}$ and $\ker(\pi \circ g) = g^{-1}(\ker \pi)$. Thus for any $Q \in \text{Prim}(C)$, there is a unique $\bar{Q} \in \text{Prim}(B)$ with $\bar{Q} \supset A$, $g(\bar{Q}) = Q$. If $\pi \in \hat{A}$ then there is a unique $\pi \in \hat{B}$ such that $\bar{\pi}$ extends π, $H_{\bar{\pi}} = H_\pi$ (the representing Hilbert spaces), and $\pi(A)$ is dense in $\bar{\pi}(B)$ in the strong operator topology of $B(H_\pi)$ (see [2, §2.11.2, p. 52]). If we let $P = \ker \pi$ and $\bar{P} = \ker \bar{\pi}$, we see that each element P of Prim (A) extends uniquely to an element $\bar{P}$ of Prim (B). Let Prim $^A(B) = \{\bar{P} \mid P \in \text{Prim}(A)\}$ and Prim $_A(B) = \{\bar{Q} \mid Q \in \text{Prim}(C) = \text{Prim}(B/A)\}$. Then Prim $^A(B)$ is homeomorphic to Prim (A) and is an open subset of Prim (B), and Prim $_A(B)$ (the complement of Prim $^A(B)$ in Prim (B)) is a closed subset of Prim (B) homeomorphic with Prim (C) (see [2, §3.2.1, p. 61]).

We will sometimes identify Prim (A) with Prim ${}^{A}(B)$ and Prim (C) with Prim ${}_{A}(B)$. We now give a characterization of $\bar{P}$ for $P \in$ Prim (A).

6.1. Lemma. *If* $P \in$ Prim (A), *then* $\bar{P}=\{x \in B \mid xA+Ax \subset P$ *for all* $a \in A\}$.

Proof. See [10, p. 206].

Throughout the remainder of this discussion, the algebras A, B, and C and the corresponding extension will all be fixed. If $P \in$ Prim (A) or $Q \in$ Prim (C), then the corresponding elements of Prim (B) will always be denoted $\bar{P}$ and $\bar{Q}$ respectively. The ideal in Prim $(M(A))$ which corresponds to P will be denoted $\tilde{P}$. We will let A° denote as before the annihilator of A in B and we will let $\gamma\colon C \to O(A)$ be the $*$-homomorphism corresponding to our fixed extension. The following diagram is then commutative

$$\text{(D)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\ & & {\scriptstyle I_A}\downarrow & & {\scriptstyle \mu}\downarrow & & {\scriptstyle \gamma}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\ \mu_0\ } & M(A) & \xrightarrow{\ \pi\ } & O(A) & \longrightarrow & 0 \end{array}$$

(see diagram in Theorem 4.3 for notation). We now remark that it is immediate from Lemma 6.1 that $\bigcap_{P \in \mathrm{Prim}(A)} \bar{P}=A^{\circ}$.

6.2. Proposition. Prim (A) *is dense in* Prim $(M(A))$, *where we are again identifying* Prim (A) *and* Prim ${}^{A}(M(A))$.

Proof. By the definition of the Jacobson topology, we need only show that $\bigcap_{P \in \mathrm{Prim}(A)} \tilde{P}=0$. This follows immediately from the above remark if we just notice that $M(A)$ is an extension of A and the annihilator of A in $M(A)$ is zero by Proposition 3.7(ii).

6.3. Theorem. *The closure of* Prim (A) *in* Prim (B), *again making the obvious identifications, is* Prim $(A) \cup \{\bar{Q} \mid Q \in$ Prim (C) and $Q \supset \ker \gamma\}$.

Proof. An ideal $\bar{Q}$, where $Q \in$ Prim (C), lies in the closure of Prim (A) exactly when it contains the intersection of all $\bar{P}$, $P \in$ Prim (A), which is A° by the above remark. Referring to diagram (D), we have $A^{\circ}=\ker \mu$, and $A=\mu(A)=\ker \pi$ (where we identify A with its image in both B and $M(A)$). Therefore $A+A^{\circ}=\ker \pi\mu$ $=\ker \gamma g=g^{-1}(\ker \gamma)$, and so $\bar{Q} \supset A^{\circ}$ if and only if $\bar{Q} \supset A+A^{\circ}$, which is so if and only if $Q \supset g(A+A^{\circ})=\ker \gamma$. Q.E.D.

6.4. Corollary. Prim (A) *is dense in* Prim (B) *if and only if* γ *is injective.*

We now discuss the separation properties of Prim(B) in terms of γ. If $P \in$ Prim(A), then $A+\tilde{P}$ is a closed ideal in $M(A)$ ([2, p. 18, 1.8.4]). Define $K_A=\bigcap_{P \in \mathrm{Prim}(A)}(A+\tilde{P})$. Then K_A is a closed, two-sided ideal in $M(A)$. Let $\bar{K}_A=\pi(K_A) \subset O(A)$.

6.5. THEOREM. Prim (B) *is* T_1 *if and only if the following two conditions are satisfied.*

(i) Prim (A) *and* Prim (C) *are* T_1;

(ii) $\gamma(C) \subset \bar{K}_A$.

Proof. The T_1 condition is equivalent with points being closed, which in the Jacobson topology means that all primitive ideals are maximal. Now Prim (C) is closed in Prim (B) (making the usual identifications), so if points in Prim (C) are closed, they are closed in Prim (B). Now to prove the theorem, we first suppose that Prim (B) is T_1. Clearly (i) must be satisfied. If $\gamma(C) \not\subset \bar{K}_A$, then $\gamma^{-1}(\bar{K}_A)$ is a proper ideal in C and so is contained in some $Q_0 \in$ Prim (C) (see [2, §2.9.7]). Let $c \notin Q_0$ and $m \in \pi^{-1}\gamma(c)$. Then (m, c) is in B (recall the construction of B) and in fact is in $\bar{Q}_0$, but $m \notin K_A$. Thus there is an ideal P_0 in Prim (A) such that $m \notin A + \tilde{P}_0$, and so $m \notin \tilde{P}_0$. It is immediate from the definitions and Lemma 6.1 that if $m \notin \tilde{P}_0$, then $(m, c) \notin \bar{P}_0$, and so we have that if $(m, c) \in \bar{P}_0$, $c \in Q_0$ and therefore $(m, c) \in \bar{Q}_0$. Thus $\bar{P}_0 \subset \bar{Q}_0$, and $\bar{P}_0$ is not closed in Prim (B). Conversely suppose that (i) and (ii) are satisfied. Since (i) holds, Prim (C) is T_1 and by the remark made above, all points $\bar{Q}_0$, $Q_0 \in$ Prim (C), are closed in Prim (B). Now let $P_0 \in$ Prim (A), and $Q_0 \in$ Prim (C) be fixed. Choose $c \notin Q_0$ and $m \in \pi^{-1}\gamma(c)$. Then $(m, c) \in B$ and by condition (ii), $m \in K_A$. Thus $m \in A + P_0$ and for some $a \in A$, $m - a \in P_0$. We therefore have $(m-a, c) \in \bar{P}_0$ but $(m-a, c) \notin \bar{Q}_0$. Hence $\bar{P}_0 \not\subset \bar{Q}_0$, and $\bar{P}_0$ is closed in Prim (B). Since every point of Prim (B) is either of the form $\bar{P}_0$ or $\bar{Q}_0$, the proof is completed.

We mention at this time the following important unsolved problem. Find conditions on γ such that if Prim (A) and Prim (C) are Hausdorff, so is Prim (B).

7. **An example.** Let N denote the positive integers and M_2 the algebra of all two-by-two complex valued matrices. One of the simplest noncommutative C^*-algebras is the algebra of all sequences of two-by-two, complex valued matrices which converge to a multiple of the identity. This algebra is an extension of the algebra $A = C_\infty(N, M_2)$ of all sequences in M_2 which converge to the zero matrix (in the norm in M_2) by the complex numbers $\mathbf{C}$. We may think of A as $C_\infty(N) \otimes M_2$ where $C_\infty(N)$ is the algebra of all sequences of complex numbers which converge to 0. We shall investigate all such extensions. There are an infinite number of strong equivalence classes of such extensions, but we shall see that there are only seven weak equivalence classes and we shall give an explicit representation of each. We shall also see that all such extensions split (this is not obvious a priori) and we shall investigate the primitive ideal spaces.

By previous results we have $M(A) = C_b(N, M_2)$, the bounded sequences from M_2. If $\gamma \in$ Hom $(\mathbf{C}, O(A))$, and $\rho: \mathbf{C} \to M(A)$ is a linear map with $\pi\rho = \gamma$, we say that ρ is compatible with γ. Both γ and ρ are completely determined by their values at 1 and the fact that they are linear. Since γ is a $*$-homomorphism, $\gamma(1)^2 = \gamma(1^2) = \gamma(1)$ and $\gamma(1)^* = \gamma(1^*) = \gamma(1)$. Thus $\gamma(1)$ is a projection in $O(A)$. Then $(\rho(1)^* + \rho(1))/2$

projects onto $(\gamma(1)^*+\gamma(1))/2=\gamma(1)$. Similarly $\rho(1)^2$ projects onto $\gamma(1)$. We may therefore assume that $\rho(1)$ is a positive self adjoint element in $M(A)$. We then have the fact that for each $n\in N$, $\rho(1)(n)$ is a positive self adjoint matrix. We know that there is a unitary matrix U_n such that $U_n\rho(1)(n)U_n^*=D(\sigma_{11}(n), \sigma_{22}(n))$ where $D(a, b)$ represents a diagonal matrix with entries a and b on the main diagonal, and $\sigma_{11}(n)$ and $\sigma_{22}(n)$ are positive real numbers. If we define a $*$-homomorphism $\theta\colon M(A)\to M(A)$ by $(\theta x)(n)=U_nx(n)U_n^*$ for all $n\in N$, we see that $\|x(n)\|=\|U_nx(n)U_n^*\|$. This means that $\theta(A)=A$ and so θ is an extension of a $*$-isomorphism of A onto itself to a $*$-isomorphism of $M(A)$ onto itself. If we denote the resulting automorphism of $O(A)$ by $\hat{\theta}$, then $\hat{\theta}\gamma I_C^{-1}=\bar{\gamma}$ produces an extension which is weakly equivalent with that corresponding to γ. We may therefore replace γ by $\bar{\gamma}$ and so may assume that $\rho(1)(n)=D(\sigma_{11}(n), \sigma_{22}(n))$. Since $\gamma(1)^2=\gamma(1)$, we know that $\rho(1)^2-\rho(1)$ is in A. This means that

$$\lim_{n\to\infty} \|(\rho(1)(n))^2-\rho(1)(n)\| = 0$$

therefore

$$\lim_{n\to\infty} |\sigma_{ii}(n)^2-\sigma_{ii}(n)| = 0, \qquad i = 1, 2.$$

The latter holds because the norm of a diagonal matrix is the maximum absolute value of any of its entries.

The real valued function of a real variable $g(t)=t^2-t$ is a parabola vanishing at zero and one. It is easy to see that given any $\varepsilon>0$, there is a $\delta>0$ such that if $|g(t)|<\delta$ then either $|t|<\varepsilon$ or $|t-1|<\varepsilon$. Now define a linear map $\rho'\colon \boldsymbol{C}\to M(A)$ by $\rho'(1)(n)=D(\mu_{11}(n), \mu_{22}(n))$ where the μ_{ii} are defined by

$$\mu_{ii}(n) = \chi_{(1/2,\infty)}(\sigma_{ii}(n)), \qquad i = 1, 2.$$

Here χ_M is the characteristic function of the set M.

7.1. Lemma. $\rho'-\rho\in A$.

Proof. For all $n\in N$ we have

$$(\rho'(1)-\rho(1))(n) = D(\sigma_{11}(n)-\mu_{11}(n), \sigma_{22}(n)-\mu_{22}(n)).$$

Given $\varepsilon>0$, let δ be so small that whenever $|\sigma_{ii}(n)^2-\sigma_{ii}(n)|<\delta$, either $|\sigma_{ii}(n)|<\varepsilon$ or $|\sigma_{ii}(n)-1|<\varepsilon$. We may assume that $\varepsilon<1/4$ so that if $|\sigma_{ii}(n)|<\varepsilon$, $\mu_{ii}(n)=0$, and if $|\sigma_{ii}(n)-1|<\varepsilon$, $\mu_{ii}(n)=1$, $i=1, 2$. In either case $|\sigma_{ii}(n)-\mu_{ii}(n)|<\varepsilon$. Combining the above statements we see that given any $\varepsilon>0$, we may choose an integer n_0 such that $n>n_0$ implies $|\sigma_{ii}(n)^2-\sigma_{ii}(n)|<\delta$, $i=1, 2$, and thus

$$\|\rho(1)(n)-\rho'(1)(n)\| = \sup_{i=1,2} |\sigma_{ii}(n)-\mu_{ii}(n)| < \varepsilon.$$

This statement means that $\lim_{n\to\infty} \|\rho(1)(n)-\rho'(1)(n)\|=0$, and so $\rho'-\rho\in A$ and the lemma is complete.

Let e_i be the two by two matrix whose entries are δ_{ij} in the i, j position, where δ_{ij}

is the Kronecker delta. Denote the zero matrix by 0 and the identity matrix by I. Since by Lemma 6.1 we may replace ρ by ρ', we may assume that for every $n \in N$, $\rho(1)(n) \in \{0, e_1, e_2, I\}$. Since there is a unitary two by two matrix U with $Ue_2U^* = e_1$, we may use the method employed in the beginning of this example to assume that e_1 replaces e_2 whenever the latter occurs. We therefore assume that $\rho(1)(n) \in \{0, e_1, I\}$.

7.2. DEFINITION. (i) Let $S_{\rho,0} = \{n \in N \mid \rho(1)(n) = 0\}$,
(ii) let $S_{\rho,1} = \{n \in N \mid \rho(1)(n) = e_1\}$,
(iii) let $S_{\rho,2} = \{n \in N \mid \rho(1)(n) = I\}$,
(iv) define c_i to be the cardinality of $S_{\rho,i}$ $(i = 0, 1, 2)$.

If c_i is finite for any i, then for any $n \in S_{\rho,i}$ we may replace $\rho(1)(n)$ by some other element in the set $\{0, e_1, I\}$ and still have ρ compatible with γ. We may therefore assume that for each i, $c_i = 0$ or $c_i = \aleph_0$. C_0, C_1, and C_2 may assume these values independently, and so there are eight possibilities. The only one which cannot occur is $C_0 = 0$, $C_1 = 0$, $C_2 = 0$, since the sum of the cardinalities must be $\aleph_0$.

7.3. LEMMA. *Let T be a bijection of N onto itself. Then T induces a $*$-automorphism of $M(A)$, denoted by $\bar{T}$, such that $\bar{T}(A) = A$. $\bar{T}$ is defined by the equation $(\bar{T}x)(n) = x(T^{-1}(n))$, $n \in N$, $x \in A$.*

Proof. The elements of $M(A)$ are bounded functions on N. It is clear that composition with a fixed map $T: N \to N$ gives a $*$-homomorphism from $M(A)$ to itself. This $*$-homomorphism is actually an isomorphism since T is a bijection. Finally $\lim_{n\to\infty} T(n) = \infty$ and so if $x \in M(A)$ vanishes at infinity, so does $\bar{T}x$. This means that $\bar{T}(A) = A$ and the proof is complete.

Suppose now that $\gamma: \boldsymbol{C} \to M(A)$ is a $*$-homomorphism and $T: N \to N$ is a bijection with corresponding $*$-automorphism $\bar{T}$ of $M(A)$. If $\hat{T}$ is the induced $*$-automorphism of $O(A)$ and I_C is the identity map on C, then we know that γ and $\hat{T}\gamma I_C^{-1} = \gamma'$ are weakly equivalent. If we are interested in the corresponding extensions only up to weak equivalence, we may always replace γ by γ'. If $\rho: \boldsymbol{C} \to M(A)$ is a linear map compatible with γ, this corresponds to replacing ρ by $\rho' = \bar{T}\rho$.

Returning now to the classification, we suppose that ρ is such that $C_0 = 0$, $C_1 = \aleph_0$, $C_2 = \aleph_0$. We can obviously map N onto itself in a one to one fashion if we map $S_{\rho,1}$ onto the odd integers and $S_{\rho,2}$ onto the even integers. This means that we may assume $\rho(1)(2n+1) = e_1$ and $\rho(1)(2n) = I$, $n \in N$, and that ρ is compatible with γ. In all other cases where two of the c_i are $\aleph_0$, we may perform the analogous normalization. In the case where all the c_i are $\aleph_0$, we may assume that γ has a compatible ρ with $\rho(1)(3n+1) = 0$, $\rho(1)(3n+2) = e_1$, and $\rho(1)(3n) = I$. We list below, the seven extensions corresponding to the seven homomorphisms γ described above. We have shown that every extension of A by $\boldsymbol{C}$ is weakly equivalent to one of these, and it may be easily seen that no two of these extensions are equivalent. One may verify that these extensions do in fact correspond to the homomorphisms described above by showing that the maps derived from the extensions have compatible ρ's as listed above.

We will denote a two-by-two matrix X by (x_{ij}) putting in evidence the components of the matrix.

(1) Corresponding to $c_0=\aleph_0$, $c_1=0$, $c_2=0$ we have the algebra $B_1=A\times \boldsymbol{C}$, the product of the algebras A and $\boldsymbol{C}$.

(2) Corresponding to $c_0=0$, $c_1=\aleph_0$, $c_2=0$ we have the algebra

$$B_2 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(n)) = x_\infty e_1\right\}$$

where x_∞ is a complex number depending on (x_{ij}).

(3) Corresponding to the case $c_0=0$, $c_1=0$, $c_2=\aleph_0$, we have

$$B_3 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(n)) = x_\infty I\right\}.$$

(4) When $c_0=0$, $c_1=\aleph_0$, $c_2=\aleph_0$, we have

$$B_4 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(2n)) = x_\infty I \text{ and } \lim_{n\to\infty} (x_{ij}(2n+1)) = x_\infty e_1\right\}.$$

(5) When $c_0=\aleph_0$, $c_1=0$, $c_2=\aleph_0$, we have

$$B_5 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(2n)) = x_\infty I \text{ and } \lim_{n\to\infty} (x_{ij}(2n+1)) = 0\right\}.$$

(6) When $c_0=\aleph_0$, $c_1=\aleph_0$, $c_2=0$, we have

$$B_6 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(2n)) = x_\infty e_1 \text{ and } \lim_{n\to\infty} (x_{ij}(2n+1)) = 0\right\}.$$

(7) When $c_0=\aleph_0$, $c_1=\aleph_0$, $c_2=\aleph_0$, we have

$$B_7 = \left\{(x_{ij}) \in M(A) \mid \lim_{n\to\infty} (x_{ij}(3n)) = x_\infty I,\ \lim_{n\to\infty} (x_{ij}(3n+1)) = 0, \text{ and } \lim_{n\to\infty} (x_{ij}(3n+1)) = x_\infty e_1\right\}.$$

We note in passing that the primitive ideal spaces of these extensions are easily computed. Prim (B_1) is N with one point adjoined discretely. Prim (B_2), Prim (B_3), and Prim (B_4) are all equal to $\overline{N}$, the one point compactification of N. Prim (B_5), Prim (B_6), and Prim (B_7) are all equal to the space N' consisting of the disjoint union of the odd positive integers and the one point compactification of the even positive integers.

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DREXEL INSTITUTE OF TECHNOLOGY,
PHILADELPHIA, PENNSYLVANIA