On Independence

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This short note aims to give a proof of Proposition 1 below, and show how to use this proposition to determine whether a Gaussian process satisfies the independence condition for a Brownian motion. Throughout this note, whenever we need a filtration for a stochastic process, we take the natural filtration.

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let d be a positive integer, A be a non-empty set, and for each $\alpha \in A$, $X_\alpha : \Omega \to \mathbb{R}^d$ be a random vector. Then, for any random vector $X : \Omega \to \mathbb{R}^d$, the following statements are equivalent:*

1. *X is independent of $\sigma(X_\alpha | \alpha \in A)$;*
2. *X is independent of $\sigma(X_{\alpha_1}, \cdots, X_{\alpha_n})$ for any finite subset $\{ \alpha_1, \cdots, \alpha_n \}$ of A.*

1 Implication of Proposition 1

Suppose that we are given a Gaussian process $(X_t)_{t \geq 0}$, and we want to check whether $(X_t)_{t \geq 0}$ is a (one-dimensional) Brownian motion or not. Among all the axioms of a Brownian motion, it is the most technical to check whether for any $t \geq s \geq 0$, the random variable $X_t - X_s$ is independent of $\sigma(X_r | r \leq s)$. The following corollary of Proposition 1 provides a handy criterion for this independence condition.

Corollary 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(X_t)_{t \geq 0}$ be a Gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{E}(X_t) = 0$ for any $t \geq 0$. Then, the following statements are equivalent:*

1. *$X_t - X_s$ is independent of $\sigma(X_r | r \leq s)$ for any $t \geq s \geq 0$;*
2. *$\mathbb{E}((X_t - X_s)X_r) = 0$ for any $t \geq s \geq r \geq 0$.*

Proof. Suppose that the first statement is true. Then, for any $t \geq s \geq r \geq 0$, the first condition implies that $X_t - X_s$ is independent of X_r. In particular, the covariance between $X_t - X_s$ and X_r is zero, which is precisely what the second statement says.

Conversely, assuming that the second statement is true, we shall show that the first statement holds. Fix $t \geq s \geq 0$. Take distinct $r_1, \cdots, r_n \in [0, s]$ arbitrarily. Define $Y_j = X_{r_j}$ for each $j \in \{ 1, \cdots, n \}$ and let $Y = (Y_1, \cdots, Y_n)^T$. Since the covariance matrix $\mathrm{Cov}(Y)$ of Y is real symmetric, so there exists some n-by-n orthogonal matrix Q such that $Q\mathrm{Cov}(Y)Q^T$ is diagonal. Define

$\tilde{Y} = (\tilde{Y}_1, \cdots, \tilde{Y}_n)^T$ to be $QY = (\sum_{j=1}^n Q_{1,j}Y_j, \cdots, \sum_{j=1}^n Q_{n,j}Y_j)^T$. Then, we have $\mathrm{Cov}(\tilde{Y}) = Q\mathrm{Cov}(Y)Q^T$, which is diagonal. Further note that for any $k \in \{1, \cdots, n\}$,

$$\mathbb{E}((X_t - X_s)\tilde{Y}_k) = \sum_{j=1}^n Q_{k,j}\mathbb{E}((X_t - X_s)X_{r_j}) = 0,$$

where the second equality follows from the second statement we assume. Therefore, we have that the covariance matrix of $(X_t - X_s, \tilde{Y}_1, \cdots, \tilde{Y}_n)$ is diagonal. Clearly, $(X_t - X_s, \tilde{Y}_1, \cdots, \tilde{Y}_n)$ is also a Gaussian vector. Therefore, the random variables $X_t - X_s, \tilde{Y}_1, \cdots, \tilde{Y}_n$ are mutually independent. In particular, we have that $X_t - X_s$ is independent of $\sigma(\tilde{Y}_1, \cdots, \tilde{Y}_n) = \sigma(X_{r_1}, \cdots, X_{r_n})$. Since $r_1, \cdots, r_n$ are taken from $[0, s]$ arbitrarily, we deduce from Proposition 1 that $X_t - X_s$ is independent of $\sigma(X_r | r \leq s)$. □

During the lecture on Oct. 25th, a stochastic process $(Z_t)_{t\geq 0}$ is defined by $Z_0 = 0$ a.s. and $Z_t = tB_{1/t}$ a.s. for $t > 0$, where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion. It can be seen from Corollary 2 that one only needs to evaluate several expectations when she or he is to show that $(Z_t)_{t\geq 0}$ is a Brownian motion.

2 Proof of Proposition 1

To show Proposition 1, we need a technical theorem in measure theory called Dynkin's π-λ theorem. To state the theorem, we need the following definitions.

Definition. *Let Ω be a non-empty set. A π-system of Ω is a non-empty family $\mathcal{P}$ of subsets of Ω satisfying that $E_1 \cap E_2 \in \mathcal{P}$ for any $E_1, E_2 \in \mathcal{P}$. A Dynkin system of Ω, or a λ-system of Ω, is a family $\mathcal{D}$ of subsets of Ω satisfying that*

1. *$\Omega \in \mathcal{D}$;*
2. *for any $E \in \mathcal{D}$, $\Omega \setminus E \in \mathcal{D}$;*
3. *for any pairwise disjoint sequence $(E_k)_{k\in\mathbb{N}}$ of sets in $\mathcal{D}$, $\bigcup_{k\in\mathbb{N}} E_k \in \mathcal{D}$.*

Proposition 3. *Let Ω be a non-empty set. Then, a σ-algebra of Ω is both a π-system and a Dynkin system.*

Proof. Trivial. □

It is also clear that the intersection of arbitrarily many π-systems is a π-system, and the intersection of arbitrarily many Dynkin systems is a Dynkin system. Therefore, the following definition makes sense.

Definition. *Let Ω be a non-empty set, and $\mathcal{S}$ be a family of subsets of Ω. Then, the π-system generated by $\mathcal{S}$, denoted by $\pi(\mathcal{S})$, is the minimum π-system containing $\mathcal{S}$, and the Dynkin system generated by $\mathcal{S}$, denoted by $\delta(\mathcal{S})$, is the minimum Dynkin system containing $\mathcal{S}$. Equivalently, we can also define $\pi(\mathcal{S})$ and $\delta(\mathcal{S})$ by*

$$\pi(\mathcal{S}) = \bigcap_{\pi\text{-system } \mathcal{P}\supseteq\mathcal{S}} \mathcal{P}; \quad \delta(\mathcal{S}) = \bigcap_{\text{Dynkin system } \mathcal{D}\supseteq\mathcal{S}} \mathcal{D}.$$

Dynkin's π-λ theorem is stated as follows.

Theorem 4 (Dynkin's π-λ Theorem)**.** *Let Ω be a non-empty set, and $\mathcal{P}$ be a π-system of Ω. Then, $\sigma(\mathcal{P}) = \delta(\mathcal{P})$.*

The significance of Dynkin's π-λ theorem can be seen in the following way. Let $\mathcal{P}$ be a π-system of the sample space Ω. Suppose that there is a probability measure $\mathbb{P}$ on $(\Omega, \sigma(\mathcal{P}))$ that we are interested in. Now imagine that we want to check whether a property related to $\mathbb{P}$ holds for all the sets in $\sigma(\mathcal{P})$. At first, suppose that we can check that this property holds for all sets in $\mathcal{P}$. Then, we need to check whether this property remains valid when we take the countable union. It is natural to use the countable additivity of $\mathbb{P}$ for this purpose, but the countable additivity of $\mathbb{P}$ requires that the sets in the sequence are pairwise disjoint. Based on this observation, we can see that the third axiom of the Dynkin system is more compatible with the probability measure $\mathbb{P}$. What we wrote above can be made clear in the proof of the following lemma.

Lemma 5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{F}'$ and $\mathcal{P}$ be two subsets of $\mathcal{F}$, where $\mathcal{F}'$ is a σ-algebra and $\mathcal{P}$ is a π-system. Suppose that for any $E_1 \in \mathcal{F}'$ and $E_2 \in \mathcal{P}$, $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2)$. Then, $\mathcal{F}'$ and $\sigma(\mathcal{P})$ are independent.*

Proof. Define

$$\mathcal{D} = \{\, E \in \mathcal{F} \mid \forall E_1 \in \mathcal{F}',\, \mathbb{P}(E \cap E_1) = \mathbb{P}(E)\mathbb{P}(E_1) \,\}\,.$$

Clearly, $\mathcal{D} \supseteq \mathcal{P}$. Our goal is to show that $\mathcal{D} \supseteq \sigma(\mathcal{P}) = \delta(\mathcal{P})$, where the equality at the end follows from Dynkin's π-λ theorem. Hence, we only need to show that $\mathcal{D}$ is a Dynkin system, which can be easily checked. ☐

Proof of Proposition 1. Clearly, the first statement implies the second. Hence, we shall only prove that the second statement implies the first.

We want to use Lemma 5. The σ-algebra $\mathcal{F}'$ in Lemma 5 can be taken as $\sigma(X)$. Then, the π-system $\mathcal{P}$ in Lemma 5 should be defined in terms of $(X_\alpha)_{\alpha \in A}$. A natural choice is to take

$$\mathcal{P} = \pi\left(\bigcup_{\alpha \in A} \sigma(X_\alpha)\right).$$

We will see that everything goes well if we set $\mathcal{F}'$ and $\mathcal{P}$ in this way.

Firstly, let us see which sets are contained in $\mathcal{P}$. This question is answered by the following claim:

$$\mathcal{P} = \bigcup_{n=1}^{\infty} \left\{ \bigcap_{j=1}^{n} E_{\alpha_j} \;\middle|\; \forall j \in \{\, 1, \cdots, n \,\},\, \alpha_j \in A \text{ and } E_{\alpha_j} \in \sigma(X_{\alpha_j}) \right\}. \tag{1}$$

The proof of the equality above is not hard, so we omit it.

We assume that the second statement in Proposition 1 holds. Then, for any finite $\{\, \alpha_1, \cdots, \alpha_n \,\} \subseteq A$ and any $E_{\alpha_1} \in \sigma(X_{\alpha_1}), \cdots, E_{\alpha_n} \in \sigma(X_{\alpha_n})$, since $\bigcap_{j=1}^{n} E_{\alpha_j} \in \sigma(X_{\alpha_1}, \cdots, X_{\alpha_n})$, by the independence between X and the σ-algebra $\sigma(X_{\alpha_1}, \cdots, X_{\alpha_n})$, we have

$$\mathbb{P}\left(E \cap \bigcap_{j=1}^{n} E_{\alpha_j}\right) = \mathbb{P}(E) \cdot \mathbb{P}\left(\bigcap_{j=1}^{n} E_{\alpha_j}\right),$$

for any $E \in \sigma(X)$. Combining this fact with (1), we can thus deduce that for any $E \in \sigma(X)$ and any $E' \in \mathcal{P}$, $\mathbb{P}(E \cap E') = \mathbb{P}(E)\mathbb{P}(E')$. Applying Lemma 5, we have $\sigma(X)$ is independent of $\sigma(\mathcal{P})$. As $\mathcal{P} \supseteq \bigcup_{\alpha \in A} \sigma(X_\alpha)$, we have $\sigma(\mathcal{P}) \supseteq \sigma(X_\alpha | \alpha \in A)$. As a consequence, we can conclude that $\sigma(X)$ is independent of $\sigma(X_\alpha | \alpha \in A)$, which is equivalent to the first statement in Proposition 1. □

Remark 6. At the end of the proof above, we showed that $\sigma(\mathcal{P}) \supseteq \sigma(X_\alpha | \alpha \in A)$. Indeed, the opposite inclusion is also true. To see this, note that

$$\mathcal{P} = \pi\left(\bigcup_{\alpha \in A} \sigma(X_\alpha)\right) \subseteq \sigma(X_\alpha | \alpha \in A),$$

because a σ-algebra is always a π-system. Hence, we have

$$\sigma(\mathcal{P}) \subseteq \sigma(\sigma(X_\alpha | \alpha \in A)) = \sigma(X_\alpha | \alpha \in A).$$

3 Proof of Dynkin's π-λ Theorem

Now we give a proof of Dynkin's π-λ Theorem. The proof we are to give can be found in many textbooks on measure theory.

The following lemma is a converse of Proposition 3.

Lemma 7. *Let Ω be a non-empty set. Then, a Dynkin system of Ω which is also a π-system must be a σ-algebra of Ω.*

Proof. Let $\mathcal{F}$ be a family of subsets of Ω, which is both a Dynkin system and a π-system. Then, we only need to show that for any sequence $(E_n)_{n \in \mathbb{N}}$ of members in $\mathcal{F}$, $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{F}$. Define $\tilde{E}_n$ by letting $\tilde{E}_1 = E_1$ and $\tilde{E}_n = E_n \setminus (\bigcup_{k=1}^{n-1} E_k)$ for any integer $k \geq 2$. Here, we adopt the convention that the set $\mathbb{N}$ of natural numbers begins with 1.

We claim that $\tilde{E}_n \in \mathcal{F}$ for any $n \in \mathbb{N}$. When $n = 1$, $\tilde{E}_1 = E_1 \in \mathcal{F}$. Now we consider $n \geq 2$. For any positive integer $k < n$, $\Omega \setminus E_k \in \mathcal{F}$ because $\mathcal{F}$ is a Dynkin system. Hence, using the fact that $\mathcal{F}$ is also a π-system, we have

$$\tilde{E}_n = E_n \cap \left(\bigcap_{k=1}^{n-1} \Omega \setminus E_k\right) \in \mathcal{F}.$$

Therefore, $\tilde{E}_n \in \mathcal{F}$ for any $n \in \mathbb{N}$.

Clearly, $(\tilde{E}_n)_{n \in \mathbb{N}}$ is pairwise disjoint. We can thus conclude from the third axiom of Dynkin systems that $\bigcup_{n \in \mathbb{N}} E_n = \bigcup_{n \in \mathbb{N}} \tilde{E}_n \in \mathcal{F}$. □

Proof of Dynkin's π-λ Theorem. Let Ω be a non-empty set, and $\mathcal{P}$ be a π-system of Ω. On the one hand, by Proposition 3, we have $\sigma(\mathcal{P}) \supseteq \delta(\mathcal{P})$. On the other hand, by the previous lemma, in order to prove $\sigma(\mathcal{P}) \subseteq \delta(\mathcal{P})$, one only needs to show that $\delta(\mathcal{P})$ is a π-system.

For every $E \subseteq \Omega$, define

$$\mathcal{D}_E = \{ E' \subseteq \Omega \mid E' \cap E \in \delta(\mathcal{P}) \}.$$

Then, $\delta(\mathcal{P})$ being a π-system is equivalent to saying that $\mathcal{D}_E \supseteq \delta(\mathcal{P})$ for every $E \in \delta(\mathcal{P})$, so we only need to prove the latter statement.

Fix an arbitrary $E \in \delta(\mathcal{P})$. We shall show that $\mathcal{D}_E$ is a Dynkin system. It is straightforward that $\Omega \in \mathcal{D}_E$. It is also not hard to show that for any pairwise disjoint sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{D}_E$, we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}_E$. Thus, it only remains to show that for any $E' \in \mathcal{D}_E$, $\Omega \setminus E' \in \mathcal{D}_E$. Fix an arbitrary $E' \in \mathcal{D}_E$. Note that $E \in \delta(\mathcal{P})$ and $E \cap E' \in \delta(\mathcal{P})$. Since $\delta(\mathcal{P})$ is a Dynkin system, we have $E \cap (\Omega \setminus E') = E \setminus (E \cap E')$. Taking the complement on both sides gives

$$\Omega \setminus (E \cap (\Omega \setminus E')) = (\Omega \setminus E) \cup (E \cap E') = (\Omega \setminus E) \cup (E \cap E') \cup \varnothing \cup \varnothing \cup \cdots .$$

Note that the entries of the set sequence $(\Omega \setminus E, E \cap E', \varnothing, \varnothing, \cdots)$ are in $\delta(\mathcal{P})$ and pairwise disjoint. Therefore, using the fact that $\delta(\mathcal{P})$ is a Dynkin system, we have that $\Omega \setminus (E \cap (\Omega \setminus E')) \in \delta(\mathcal{P})$, which further implies that $E \cap (\Omega \setminus E') \in \delta(\mathcal{P})$. Therefore, we have $\Omega \setminus E' \in \mathcal{D}_E$ provided that $E' \in \mathcal{D}_E$.

Suppose that we manage to prove that $\mathcal{P} \subseteq \mathcal{D}_E$ for every $E \in \delta(\mathcal{P})$. Then, for any $E \in \delta(\mathcal{P})$, as $\mathcal{D}_E$ is a Dynkin system, we have that $\delta(\mathcal{P}) \subseteq \mathcal{D}_E$. As we remarked earlier, this implies that $\delta(\mathcal{P})$ is a π-system, which completes the proof. Therefore, we only need to show that $\mathcal{P} \subseteq \mathcal{D}_E$ for every $E \in \delta(\mathcal{P})$.

We want to prove that for any $E \in \delta(\mathcal{P})$ and any $E' \in \mathcal{P}$, $E' \in \mathcal{D}_E$, which is equivalent to $E \cap E' \in \delta(\mathcal{P})$ and further equivalent to $E \in \mathcal{D}_{E'}$. We may thus rewrite the statement as follows. For any $E' \in \mathcal{P}$, we want to prove that $\delta(\mathcal{P}) \subseteq \mathcal{D}_{E'}$. Since $E' \in \mathcal{P}$, it is clear that $\mathcal{P} \subseteq \mathcal{D}_{E'}$. Therefore, since $\mathcal{D}_{E'}$ is a Dynkin system, we have $\delta(\mathcal{P}) \subseteq \mathcal{D}_{E'}$. This completes our proof. $\square$