Report

Yuu Hiramatsu

Excersise 4.4.6

For $f(t,x)=tx^2$, set $I=\int_0^t sB_s\,dB_s$

From 4.4.5,

$$\int_0^t [\partial_x f](s,B_s)\,dB_s = f(t,B_t)-f(0,0) - \int_0^t \left\{[\partial_t f](s,B_s)+\frac{1}{2}[\partial_x^2 f](s,B_s)\right\}ds \cdots ☆$$

Since $\partial_x f(t,x)=2tx \Rightarrow [\partial_x f](s,B_s)=2sB_s$,

$$\int_0^t [\partial_x f](s,B_s)\,dB_s = \int_0^t 2s\,ds\,dB = 2I \cdots ①$$

And since $\partial_t f(t,x)=x^2$, $\partial_x^2 f(t,x)=2t$,

$$f(t,B_t)-f(0,0)-\int_0^t \left\{[\partial_t f](s,B_s)+\frac{1}{2}[\partial_x^2 f](s,B_s)\right\}ds$$

$$= t(B_t)^2 - 0 - \int_0^t \left\{2(B_s)^2 + \frac{1}{2}\cdot 2s\right\}ds$$

$$= t(B_t)^2 - \int_0^t (B_s)^2 ds - \frac{1}{2}t^2 \cdots ②$$

from ①, ②,

$$2I = t(B_t)^2 - \int_0^t (B_s)^2 ds - \frac{1}{2}t^2$$

$$\therefore I = \frac{t}{2}(B_t)^2 - \frac{1}{2}\int_0^t (B_s)^2 ds - \frac{1}{4}t^2$$ //

Yuu Hiramatsu

Excersise 4.4.7

$f(t,x) = e^{x-\frac{1}{2}t}$ $\quad \frac{\partial}{\partial x} f(t,x) = e^{x-\frac{1}{2}t}$, $\frac{\partial}{\partial t} f(t,x) = -\frac{1}{2} e^{x-\frac{1}{2}t}$

$\frac{\partial^2}{\partial x^2} f(t,x) = e^{x-\frac{1}{2}t}$

$$f(t,B_t) = f(0,0) + \int_0^t [\partial_x f](s,B_s)\,dB_s$$
$$+ \int_0^t \left\{ [\partial_t f](s,B_s) + \frac{1}{2}(\partial_x^2 f)(s,B_s) \right\} ds$$

$$f(t,B_t) = f(0,0) + \int_0^t \exp\left(s - \frac{B_s}{2}\right) dB_s$$
$$- \int_0^t \frac{1}{2}\exp\left(s - \frac{B_s}{2}\right) + \frac{1}{2}\exp\left(s - \frac{B_s}{2}\right) ds$$

$$f(t,B_t) = f(0,0) + \int_0^t [\partial_x f](s,B_s)\,dB_s$$

since $[\partial_x f](s,B_s) = f(s,B_s)$

$$f(t,B_t) = f(0,0) + \int_0^t f(s,B_s)\,dB_s$$

$$\underline{\int_0^t f(s,B_s)\,dB_s = f(t,B_t) - f(0,0)}\ //$$

Yan Hiramatsu

4.4.8 $f(t,x) = e^{(c-\frac{1}{2}\sigma^2)t+\sigma x}$ $c \in \mathbb{R}, \sigma > 0$

from 4.4.4

$$f(t,B_t) = f(0,0) + \int_0^t [\partial_x f](s,B_s)\,dB_s + \int_0^t \left\{[\partial_t f](s,B_s) + \frac{1}{2}[\partial_x^2 f](s,B_s)\right\} ds$$

$\partial_x f(t,x) = \sigma f(t,x)$ $\partial_t f(t,x) = (c-\frac{1}{2}\sigma^2) f(t,x)$

$\partial_x^2 f(t,x) = \sigma^2 f(t,x)$

$$X_t = X_0 + \int_0^t \sigma f(s,B_s)\,dB_s + \int_0^t \left\{ (c-\frac{1}{2}\sigma^2) f(s,B_s) + \frac{\sigma^2}{2} f(s,B_s) \right\} ds$$

$$= X_0 + \sigma\int_0^t X_s\,dB_s + c\int_0^t X_s\,ds$$

//