SML-Report (ver.2) Yua Hiramatsu 2023, 10, 22

Exercise 1.1.6. *If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and if $A, B \in \mathcal{F}$, check that*

1) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, *where* $A^c := \Omega \setminus A$,

2) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$,

3) If $A \subset B$, *then* $\mathbb{P}(A) \leq \mathbb{P}(B)$.

1) Since $A \cup A^c = \Omega$ and $A \cap A^c = \phi$,

$\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$

$\therefore \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ $\square$

2) If $A \cap B = \phi$,

$\mathbb{P}(A \cap B) = \mathbb{P}(\phi) = 0$ and

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ from Definition 1.1.5

else if $A \cap B \neq \phi$,

Since $(A \setminus (A \cap B)) \cap (A \cap B) = \phi$ from Definition 1.1.5

$$\mathbb{P}(A \setminus A \cap B) + \mathbb{P}(A \cap B) = \mathbb{P}((A \setminus A \cap B) \cup (A \cap B)) = \mathbb{P}(A).$$

By applying this argument to $B \setminus (A \cap B)$, we get

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B \setminus (A \cap B)) + \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) - \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \ \square \end{aligned}$$

3) $A \subset B \Leftrightarrow B = A \cup (B \setminus A)$

then

$$\begin{aligned} \mathbb{P}(B) - \mathbb{P}(A) &= \mathbb{P}(A \cup (B \setminus A)) - \mathbb{P}(A) \\ &= \mathbb{P}(A) + \mathbb{P}(B \setminus A) - \mathbb{P}(A \cap (B \setminus A)) - \mathbb{P}(A) \\ &= \mathbb{P}(B \setminus A) - \mathbb{P}(\phi) \\ &= \mathbb{P}(B \setminus A) \end{aligned}$$

since $A \subset B$, $\mathbb{P}(B \setminus A) \geq \mathbb{P}(\phi) = 0$ so

$\mathbb{P}(B) - \mathbb{P}(A) \geq 0$

$\Leftrightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Lemma 1.1.7 (Continuity of probability). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$. If*
4) *$A_j \subset A_{j+1}$ for any j, then*

$$\mathbb{P}\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \lim_{j\to\infty} \mathbb{P}(A_j),$$

5) *while if $A_j \supset A_{j+1}$ for any j, then*

$$\mathbb{P}\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = \lim_{j\to\infty} \mathbb{P}(A_j).$$

4) Since $A_j \subset A_{j+1}$, $\bigcup_{j=1}^{n} A_j = A_n$.
Set C_j as follows:

$$C_j = A_j \setminus A_{j-1}, \quad C_1 = A_1.$$

By observing that

for any $j, k \in \mathbb{N}$, $j \neq k$, $C_j \cap C_k = \phi$

and

$$\bigcup_{j=0}^{n} C_j = A_n$$

We get

$$\mathbb{P}(A_n) = \mathbb{P}\Big(\bigcup_{j=1}^{n} C_j\Big) = \sum_{j=1}^{n} \mathbb{P}(C_j)$$

then

$$\mathbb{P}\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \mathbb{P}\Big(\lim_{n\to\infty}\bigcup_{j=1}^{n} A_j\Big) = \mathbb{P}\Big(\lim_{n\to\infty}\bigcup_{j=1}^{n} C_j\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\bigcup_{j=1}^{n} C_j\Big) = \lim_{j\to\infty} \mathbb{P}(A_j)$$

□

5) Since $A_j \supset A_{j+1}$, $\bigcap_{j=1}^{n} A_j = A_n$.
Set C_j as follows.

$$C_j = A_j \setminus A_{j+1} \qquad \text{for } j \geq 1$$

By observing that

for any $j, k \in \mathbb{N}$, $j \neq k$, $C_j \cap C_k = \phi$

and

$$\mathbb{P}\Big(\bigcup_{j=1}^{n-1} C_j\Big) = \mathbb{P}(A_{n-1}\setminus A_n) + \mathbb{P}(A_{n-2}\setminus A_{n-1}) + \cdots + \mathbb{P}(A_2\setminus A_3) + \mathbb{P}(A_1\setminus A_2)$$
$$= \mathbb{P}(A_1) - \mathbb{P}(A_n)$$

we get

$$\mathbb{P}\Big(\bigcap_{j=1}^{n} A_n\Big) = \mathbb{P}(A_1) - \mathbb{P}\Big(\bigcup_{j=1}^{n-1} C_j\Big) = \mathbb{P}(A_1) - \sum_{j=1}^{n-1} \mathbb{P}(C_j)$$

then

$$\mathbb{P}\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = \mathbb{P}(A_1) - \mathbb{P}\Big(\lim_{n\to\infty}\bigcup_{j=1}^{n} C_j\Big) = \mathbb{P}(A_1) - \lim_{n\to\infty}\mathbb{P}\Big(\bigcup_{j=1}^{n} C_j\Big) = \lim_{j\to\infty}\mathbb{P}(A_j)$$

□