The Gambler's Ruin Problem

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1 Introduction: a fair coin toss

Let's imagine that two people A and B are playing a coin tossing game, where the probability of the coin landing heads or tails is both 50%. A has a yen, and B has b yen (of course, both a and b are positive integers). Each side bets 1 yen in each round. The game stops only when A or B loses all their capital. So, what are the winning probabilities for A and B?

Taking A as an example, let's assume that after several rounds of the game, A has n yen. Then, in the next round, the capital in A's hand will only become $n-1$ or $n+1$ with a probability of 50%: 50%. Let $P(i)$ denote the probability that the gambler wins when he has i yen at the beginning. Clearly, $P(0) = 0$ and $P(X) = 1$ (the sum of their money is X yen) by definition. Then:

$$P(n) = 0.5 \cdot P(n-1) + 0.5 \cdot P(n+1)$$

Let's rearrange the equation:

$$P(n) - P(n-1) = P(n+1) - P(n) \quad \text{(arithmetic progression)}$$

For A's outcome, it's either losing all (0) or winning all of B's $a+b$. If we solve the above arithmetic progression, we get:

The probability of A losing all is: $\frac{b}{a+b}$

The probability of B losing all is: $\frac{a}{a+b}$

Let's verify with the example where A has 100 yen and B has 20 yen. In this case, the probability of A losing all is $\frac{20}{120} = 16.66\%$.

I did an experiment with 1000 trials using python, and the result was precise.

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When A has 100 yen and B has 20 yen, the probability of A lossing all is: 16.80%
When A has 100 yen and B has 20 yen, the probability of B lossing all is: 83.20%
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When B has 100 yen, the probability of A losing all is $\frac{100}{200} = 50\%$.

When B has 1000 yen, the probability of A losing all is $\frac{1000}{1100} = \frac{10}{11} = 90.9\%$.

Similarly, I did another two experiments with these two cases, and the results were accurate:

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When A has 100 yen and B has 100 yen，the probability of A lossing all is：50.50%
When A has 100 yen and B has 100 yen，the probability of B lossing all is：49.50%
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When A has 100 yen and B has 1000 yen，the probability of A lossing all is：90.40%
When A has 100 yen and B has 1000 yen，the probability of B lossing all is：9.60%
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2 For an arbitrary probability p

Now we change the situation: A has some money, and he wants to achieve a target capital of X yen through a series of games. In each single game, A has a probability q of losing money, and the situation transits from $P(n)$ to $P(n-1)$, and is a probability p of transitioning from $P(n)$ to $P(n+1)$. Of course, $p+q=1$. Here, we specify that p is not equal to q (or equivalently, $p \neq 0.5$) to prevent the occurrence of a zero denominator in later calculations. And of course, this case has been discussed in section 1.

We have

$$P(n) = q \cdot P(n-1) + p \cdot P(n+1)$$

Solving the above formula gives:

$$P(n+1) - P(n) = \frac{q}{p}(P(n) - P(n-1))$$

This indicates that

$$P(i+1) - P(i) = \left(\frac{q}{p}\right)^i P(1), \quad 0 < i < X,$$

which means ($p \neq q$ as we have specified)

$$P(i+1) = P(1) + P(1)\sum_{k=1}^{i}\left(\frac{q}{p}\right)^k = P(1)\sum_{k=0}^{i}\left(\frac{q}{p}\right)^k = \frac{P(1)\left(1-\left(\frac{q}{p}\right)^{i+1}\right)}{1-\frac{q}{p}};$$

Let's remember the fact (by definition) that $P(X) = 1$, so

$$1 = P(X) = \frac{P(1)\left(1-\left(\frac{q}{p}\right)^{X}\right)}{1-\frac{q}{p}};$$

From this we gets the value of $P(1)$,

$$P(1) = \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^X};$$

and the value of $P(k)$ for an arbitrary k between 1 and X.

$$P(k) = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^X}$$

This formula looks elegant and beautiful, reflecting the beauty of mathematics.