# Cantelli's inequality, an improved version of Chebyshev's inequality

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January 2024

## 1 Cantelli's inequality

Let $X$ be a real random variable, $a > 0$, and $V(X) = E\Big((X - E(X))^2\Big)$. Then

$$P\Big(\Big|X - E(X)\Big| \geq a\Big) \leq \frac{2V(X)}{V(X) + a^2}$$

## 2 Proof

Let's start by considering the random variable $Y = X - E(X) + t$ for $t \geq 0$. We have

$$X - E(X) \geq a \iff X - E(X) + t \geq a + t$$

and since $a + t \geq 0$, we have

$$(X - E(X) + t)^2 \geq (a + t)^2.$$

Thus we deduce

$$P(X - E(X) \geq a) \leq P\Big((X - E(X) + t)^2 \geq (a + t)^2\Big) \leq \frac{E\Big((X - E(X) + t)^2\Big)}{(a + t)^2}$$

(here we have applied Markov's inequality).

Furthermore,

$$E\Big((X - E(X) + t)^2\Big) = E\Big((X - E(X))^2 + 2tE\Big(X - E(X)\Big) + t^2\Big) = V(X) + t^2.$$

We have thus demonstrated that

$$P(X - E(X) \geq a) \leq \frac{t^2 + V(X)}{(a + t)^2}.$$

We seek the value of $t$ for which the right-hand side of the preceding inequality is minimized. For this, we set, for $t \geq 0$,

$$f(t) = \frac{t^2 + V(X)}{(a+t)^2}.$$

The function $f$ is differentiable on $[0, +\infty)$, and on this interval,

$$f'(t) = \frac{2t(a+t)^2 - 2(a+t)\Big(t^2 + V(X)\Big)}{(a+t)^4} = \frac{2\Big(at - V(X)\Big)}{(a+t)^3}.$$

The function $f$ reaches its minimum at $t = \frac{V(X)}{a}$, and at this point, it is equal to

$$f\Big(\frac{V(X)}{a}\Big) = \frac{\frac{V(X)^2}{a^2} + V(X)}{\Big(\frac{V(X)}{a} + a\Big)^2} = \frac{\frac{V(X)}{a}\Big(\frac{V(X)}{a} + a\Big)}{\Big(\frac{V(X)}{a} + a\Big)^2} = \frac{V(X)}{V(X) + a^2}.$$

Based on this minimum value, we apply the preceding inequality to $T = -X$ (this has the same variance as $X$), and then we obtain

$$P\Big( - \Big(X - E(X)\Big) \geq a\Big) \leq \frac{V(X)}{V(X) + a^2}.$$

Since $|X - E(X)| \geq a = ((X - E(X)) \geq a) \cup (-(X - E(X)) \geq a)$, we now have

$$P(|X - E(X)| \geq a) \leq \frac{2V(X)}{V(X) + a^2}.$$

This inequality is better than Chebyshev's inequality if and only if

$$\frac{2V(X)}{V(X) + a^2} \leq \frac{V(X)}{a^2} \iff V(X) \geq a^2 \quad (\text{or } V(X) = 0).$$