Exercise 3.2.23 (Ref 1 pg 93)

4.5. **Gaussian conditioning.** We consider the Gaussian vector (X_1, X_2, X_3) with mean 0 and covariance matrix

$$\mathcal{C} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Is the vector nondegenerate?
(b) Argue that X_3 is independent of X_2 and X_1.
(c) Compute $\mathbf{E}[X_2|X_1]$. Write X_2 as a linear combination of X_1 and a random variable independent of X_1.

(a) Vector is nondegenerate because

$$\det(C) = 1 \det\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} = 8-4=4$$

(b) For Gaussian vectors, they are independent if the covariance matrix is diagonal. (Proposition 2.10, ref 1)

Since $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ are diagonal, X_3 is independent of X_1 and X_2.

Example 4.4 (Conditional expectation for Gaussian vectors. I). Let (X, Y) be a Gaussian vector of mean 0. Then

$$\mathbf{E}[Y|X] = \frac{\mathbf{E}[XY]}{\mathbf{E}[X^2]} X. \tag{4.5}$$

(c) $E(X_2 | X_1) = \frac{E(X_2 X_1)}{E(X_1^2)} X_1 = \frac{2}{2} X_1 = X_1$

X_2 as linear combination of X_1 and a random variable independent of X_1: $X_2 = (X_2 - X_1) + X_1$

$X_2 - X_1$ and X_1 are independent since $X_2 - X_1$ and X_1 are Gaussian and $\mathrm{cov}(X_2 - X_1, X_1) = \mathrm{cov}(X_2, X_1) - \mathrm{cov}(X_1, X_1) = 2 - 2 = 0$

(d) Compute $\mathbf{E}[e^{aX_2}|X_1]$ for any $a \in \mathbb{R}$. Use this to determine the conditional distribution of X_2 given X_1.

(e) Find (Z_1, Z_2, Z_3) IID standard Gaussians that are linear combinations of (X_1, X_2, X_3).

(d) $E(e^{aX_2} | X_1) = E(e^{aX_1 + a(X_2 - X_1)} | X_1) =$

$E(e^{aX_1} e^{a(X_2-X_1)} | X_1) = e^{aX_1} E(e^{a(X_2-X_1)} | X_1) =$

$e^{aX_1} E(e^{a(X_2-X_1)})$

We can recognize $E(e^{a(X_2-X_1)})$ as MGF of $X_2 - X_1$.

Mean of $X_2 - X_1$ is 0 because mean of X_2 and X_1 are both 0.

$var(X_2 - X_1) = var(X_2) + var(X_1) - 2\,cov(X_2, X_1) = 4 + 2 - 2 \cdot 2 = 2$

MGF of Gaussian variable X with mean μ and variance σ^2 is given as

$E(e^{aX}) = e^{a\mu + \frac{1}{2}a^2\sigma^2}$, Therefore $E(e^{a(X_2-X_1)}) = e^{a\cdot 0 + \frac{1}{2}a^2 \cdot 2} = e^{a^2}$

If we compute $E(e^{aX_2} | X_1) = E(e^{aX_1 + aX_2 - aX_1} | X_1) = E(e^{aX_1} e^{a(X_2-X_1)} | X_1)$

We have seen that $X_2 - X_1$ is independent of X_1 from in part (c). Therefore,

$E(e^{aX_1} e^{a(X_2-X_1)} | X_1) = e^{aX_1} E(e^{a(X_2-X_1)}) = e^{aX_1} e^{a^2} = e^{aX_1 + \frac{1}{2}\cdot 2 \cdot a^2}$

$e^{aX_1 + \frac{1}{2}\cdot 2 \cdot a^2}$ defines the MGF of a Gaussian variable mean with $\mu = X_1$, variance $\sigma^2 = 2$

which implies conditional distribution of X_2 given X_1 has mean X_1 and variance 2

(e) We use Gram-Schmidt:

Firstly, we define $Z_1 = \frac{X_1}{\sqrt{2}}$ Z_1 is a standard Gaussian.

Define $Z_2' = X_2 - E(X_2 Z_1) Z_1$, (Z_1, Z_2') is Gaussian because it is a linear transformation of (X_1, X_2)

Z_2' is uncorrelated with Z_1 because $E(Z_2' Z_1) = E(X_2 Z_1) - E(X_2 Z_1) E(Z_1^2) = 0$

which proves Z_1 and Z_2' are independent. (Ref 1 pg 31)

$$Z_2' = X_2 - E\left(X_2 \frac{X_1}{\sqrt{2}}\right) \frac{X_1}{\sqrt{2}} = X_2 - \frac{1}{2} E(X_2 X_1) X_1 = X_2 - \frac{1}{2} 2X_1 = X_2 - X_1$$

$$Var(X_2 - X_1) = var(X_2) + var(X_1) - 2\, cov(X_2, X_1) = 4 + 2 - 2 \cdot 2 = 2$$

$Z_2 = \frac{X_2 - X_1}{\sqrt{2}} = \frac{X_2}{\sqrt{2}} - \frac{X_1}{\sqrt{2}}$. Z_2 is a standard Gaussian independent of Z_1.

Similarly,

$$Z_3 = X_3 - E(X_3 Z_2) Z_2 - E(X_3 Z_1) Z_1 = X_3 - E\left(X_3\left(\frac{X_2 - X_1}{\sqrt{2}}\right)\right)\left(\frac{X_2 - X_1}{\sqrt{2}}\right) - E\left(X_3 \frac{X_1}{\sqrt{2}}\right)\frac{X_1}{\sqrt{2}} =$$

$$X_3 - \frac{1}{2}\left(E(X_3 X_2) - E(X_3 X_1)\right)(X_2 - X_1) - \frac{1}{2} E(X_3 X_1) X_1 = X_3.$$

X_3 is a standard Gaussian because its standard deviation is 1 (apparent from covariance matrix) and mean 0.

Therefore, $(Z_1, Z_2, Z_3) = \left(\frac{X_1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(X_2 - X_1), X_3\right)$