Exercise 2.1.2

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Exercise 2.1.2. *Check that if X_1, X_2 are independent and standard Gaussian random variables, then $(X_1, X_2)^T$ is a Gaussian vector. Show that the random variable $a_1X_1 + a_2X_2$ is a Gaussian random variable with mean 0 and variance $a_1^2 + a_2^2$. Generalize your result for N independent and standard Gaussian random variables.*

To prove $X = (X_1, X_2)^T$ is a Gaussian vector, we show that for $a = (a_1, a_2)^T \in \mathbb{R}^2$ $a \cdot X = a_1X_1 + a_2X_2$ is a Gaussian random variable.

PDF of Standard Gaussian random variable X:

$$\pi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

MGF of standard Gaussian random variable X:

$$t \to E(e^{tX})$$

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} \frac{e^{tx - \frac{1}{2}x^2}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x^2 - 2tx + t^2 - t^2)}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{t^2}{2}}}{\sqrt{2\pi}} dx =$$

$e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x-t)^2}}{\sqrt{2\pi}} dx$ Because $\frac{e^{-\frac{1}{2}(x-t)^2}}{\sqrt{2\pi}}$ is PDF of Gaussian random variable with mean t,

$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x-t)^2}}{\sqrt{2\pi}} dx = 1$. $E(e^{tX}) = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x-t)^2}}{\sqrt{2\pi}} dx = e^{\frac{t^2}{2}} \cdot 1 = e^{\frac{t^2}{2}}$

Define a Gaussian variable Y with mean μ, and variance σ^2

$Y = \mu + \sigma X$ where X is a standard Gaussian random variable and $\mu \in \mathbb{R}$ $\sigma > 0$ Using the previous result, the MGF of Y:

$t \to E(e^{tY}) = E(e^{t(\mu + \sigma X)}) = E(e^{t\mu} e^{\sigma t X}) = e^{t\mu} E(e^{\sigma t X}) = e^{t\mu} \cdot e^{\frac{\sigma^2 t^2}{2}} = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$ Since MGF uniquely determines probability distribution of random variable

we can see MGF with the form $t \to e^{t\mu + \frac{1}{2}\sigma^2 t^2}$ is of a Gaussian distribution with mean μ and variance σ^2.

MGF of standard Gaussian variable X_1: $t \to E(e^{tX_1}) = e^{\frac{t^2}{2}}$

MGF of standard Gaussian variable X_2: $t \to E(e^{tX_2}) = e^{\frac{t^2}{2}}$

MGF of $a_1X_1 + a_2X_2$: $t \to E(e^{t(a_1X_1 + a_2X_2)}) = E(e^{a_1 t X_1} \cdot e^{a_2 t X_2}) \underset{\text{Independence of } X_1 \text{ and } X_2}{\overset{\text{Proposition 1.30 (Ref 2)}}{=}} E(e^{a_1 t X_1}) E(e^{a_2 t X_2}) = e^{\frac{a_1^2 t^2}{2}} e^{\frac{a_2 t^2}{2}} = e^{\frac{a_1^2 + a_2^2}{2} t^2}$

$t \to e^{\frac{a_1^2 + a_2^2}{2} t^2}$ defines the MGF of Gaussian random variable with mean 0 and variance $\frac{a_1^2 + a_2^2}{2}$

Generalization:

Let $X_1, \ldots, X_N$ be independent and standard Gaussian random variables and $a = (a_1, a_2, \ldots, a_N)^T \in \mathbb{R}^N$

MGF of $a_1X_1 + a_2X_2 + \ldots + a_NX_N$: $t \to E(e^{t(a_1X_1 + \ldots + a_NX_N)}) = E(e^{a_1 t X_1} e^{a_2 t X_2} \ldots e^{a_N t X_N}) = E(e^{a_1 t X_1}) E(e^{a_2 t X_2}) \ldots E(e^{a_N t X_N}) =$

$e^{\frac{1}{2}(a_1^2 + a_2^2 + \ldots + a_N^2) t^2}$. Thus $t \to e^{\frac{1}{2}(a_1^2 + \ldots + a_N^2) t^2}$ defines the MGF of Gaussian random variable with mean 0 and variance $\frac{a_1^2 + a_2^2 + \ldots + a_N^2}{2}$