13.3 (p. 618) In a standard Black–Scholes model let us consider an investor that constructs a self-financing portfolio V with the constraint that the value of the component invested into the risky asset is constant and equal to some value $M > 0$.

a) Assuming that the value V_0 at time 0 is deterministic and positive, compute the distribution of V_t and compute $\mathrm{E}[V_t]$, as a function of V_0 and of the parameters r, μ, σ of the Black–Scholes model.
b) Is V admissible?

a) self-financing : $dV_t = H_0(t)dS_0(t) + H_1(t)dS_1(t)$ ... ①

where $H_1(t) = M/S_1(t)$ ... ②

Define $V_t = H_0(t)S_0(t) + H_1(t)S_1(t)$ ... ⊛

$= H_0(t)S_0(t) + M$ (by ②) ... $(*)$

Then $\underset{(\text{by } ①)}{dV_t} = H_0(t)\,dS_0(t) + H_1(t)\,dS_1(t) = H_0(t)dS_0(t) + S_0(t)\,dH_0(t)$ (by $(*)$)

thus $dH_0(t) = \frac{H_1(t)}{S_0(t)}dS_1(t) = \frac{M}{S_0(t)S_1(t)}dS_1(t) = \frac{Me^{-rt}}{S_1(t)}dS_1(t)$

By BSM, we have $dH_0(t) = Me^{-rt}(b\,dt + \sigma dB_t)$

then $H_0(t) = H_0(0) + \frac{Mb}{r}(1-e^{-rt}) + \sigma M\int_0^t e^{-rs}\,dB_s$ ... (⊛⊛)

Finally, by (⊛) & (⊛⊛), we have

$V_t = H_0(t)S_0(t) + H_1(t)S_1(t) = H_0(t)S_0(t) + M$

$= e^{rt}\left(H_0(0) + \frac{Mb}{r}(1-e^{-rt}) + \sigma M\int_0^t e^{-rs}\,dB_s\right) + M$

$$E(V_t) = e^{rt}\left(H_0(0) + \frac{\mu b}{r}(1-e^{-rt})\right) + M$$
$$= e^{rt}\left(V_0 - M + \frac{\mu b}{r}(1-e^{-rt})\right) + M \qquad (\text{by } (\star\star\star))$$

Note: $V_t = H_0(t) + S_0(t) + M$ (by $(*)$)
$$V_0 = H_0(0) + S_0(0) + M$$
$$H_0(0) = V_0 - M \quad \ldots (\star\star\star)$$

b) No, since V_t has a Normal Distribution with a finite variation, thus $V_t < 0$ is possible which is not admissible.

Exercise 11.17: Assume that $S(T)/S$ does not depend on S, where $S(T)$ is the price of stock at T and $S = S(0)$. Let T be the exercise time and K the exercise price of the call option. Show that the price of this option satisfies the following PDE:

$$C = S\frac{\partial C}{\partial S} + K\frac{\partial C}{\partial K}.$$

You may assume all the necessary differentiability. Hence show that the delta of the option $\frac{\partial C}{\partial S}$ in the Black–Scholes model is given by $\Phi(h(t))$ with $h(t)$ given by (11.36).

Define $\ell(v) = e^{-rT} E((S_T/S - v)^+)$,

it equal to the payoff function of the call option when $v = K/S$, i.e. $C = e^{-rT} E((S_T - K)^+) = S\ell(K/S)$

then $\partial C/\partial S = \ell(K/S) - K\ell'(K/S)/S$

$\partial C/\partial K = \ell'(K/S)$

therefore, $S(\partial C/\partial S) + K(\partial C/\partial K)$

$= S\ell(K/S) - K\ell'(K/S) + K\ell'(K/S)$

$= S\ell(K/S) = C$

Derivation of Black-Schole Model : Expectation (Call Option)

Intuition : Option price = discounted value of expected payoff

$$\Rightarrow C_0 = e^{-rT} E\left[(S_T - K)^+\right], \text{ where } \begin{cases} S_T = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T} \cdot z} \\ z = \left(\ln(S_T/S_0) - (r - \sigma^2/2)T\right)/(\sigma\sqrt{T}) \end{cases}$$

And $(S_T - K)^+ \sim N(0, 1^2)$

Consider the case of "In Money",

$$\Leftrightarrow S_T - K \geq 0$$

$$\Leftrightarrow S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T} \cdot z} \geq K$$

$$z \geq \left(\ln(K/S_0) - (r - \sigma^2/2)T\right)/(\sigma\sqrt{T})$$

$$= \left(\ln(K/S_0 e^{rT}) + (1/2)\sigma^2 T\right)/(\sigma\sqrt{T})$$

$$= -\left(\ln(S_0 e^{rT}/K) - 1/2\,\sigma^2 T\right)/(\sigma\sqrt{T}) = -d_2 \quad \ldots (*)$$

$$\left(\text{Recall: } \mathbb{E}(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot x \cdot e^{-(x-\mu)^2/(2\sigma^2)}\, dx, \text{ for } x \sim N(\mu, \sigma^2)\right)$$

Then,

$$\mathbb{E}\left((S_T - K)^+\right) = \int_{\underset{\text{in money}}{-d_2}}^{\infty} \left(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T} \cdot x} - K\right) \cdot \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx + \underbrace{\int_{-\infty}^{-d_2} 0\, dx}_{\text{out of money}}$$

$$= e^{rT} \int_{-d_2}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) S_0\, e^{-\frac{1}{2}(\sigma^2 T - 2\sigma\sqrt{T} \cdot x + x^2)}\, dx - K \int_{\underset{\text{symm. prop. of normal distri.}}{-\infty}}^{d_2} \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx$$

$$= e^{rT} \int_{-d_2}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) S_0\, e^{-1/2(x - \sigma\sqrt{T})^2}\, dx - K \int_{-\infty}^{d_2} \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx$$

$\left(\text{Recall: } \Phi(x) = \int_{-\infty}^{x} \left(\frac{1}{2\pi}\right) e^{-x^2/2}\, dx\right)$

let $u = x - \sigma\sqrt{T}$

$dx = du$

$$= \left(e^{rT}\right)\left(S_0\right)\int_{-d_2-\sigma\sqrt{T}}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}(u)^2} du - K\,\Phi(d_2)$$

symm. prop. of normal distri.

$$= \left(e^{rT}\right)\left(S_0\right)\int_{-\infty}^{d_1} \left(\frac{1}{2\pi}\right) e^{-u^2/2}\, du - K\,\Phi(d_2)$$

$$= S_0\, e^{rT}\, \Phi(d_1) - K\,\Phi(d_2)$$

$$\Rightarrow C_0 = e^{-rT}\,\mathbb{E}\left(\left(S_T - K\right)^+\right)$$

$$= S_0\,\Phi(d_1) - K e^{-rT}\,\Phi(d_2)$$

Note: d_1 & d_2 are constant, $d_1 = d_2 + \sigma\sqrt{T}$

Derivation of Black-Schole Model : Put Option

Intuition : Option price = discounted value of expected payoff

$\Rightarrow V_0 = e^{-rT} E\left[(K - S_0)^+\right]$, where $\begin{cases} S_T = S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}\cdot z} \\ z = \left(\ln(S_T/S_0) - (r - \sigma^2/2)T\right)/(\sigma\sqrt{T}) \end{cases}$

And $(S_T - K)^+ \sim N(0, 1^2)$

Consider the case of "In Money",

$\Leftrightarrow K - S_0 \geq 0$

$\Leftrightarrow S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}\cdot z} \leq K$

$$z \leq \left(\ln(K/S_0) - (r - \sigma^2/2)T\right)/(\sigma\sqrt{T})$$

$$= \left(\ln(K/S_0 e^{rT}) + (1/2)\sigma^2 T\right)/(\sigma\sqrt{T})$$

$$= -\left(\ln(S_0 e^{rT}/K) - \tfrac{1}{2}\sigma^2 T\right)/(\sigma\sqrt{T}) = -d_2 \quad \ldots (*)$$

(Recall: $\mathbb{E}(x) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \cdot x \cdot e^{-(x-\mu)^2/(2\sigma^2)}\, dx$, for $x \sim N(\mu, \sigma^2)$)

Then,

$$\mathbb{E}\left((K - S_T)^+\right) = \underbrace{\int_{-\infty}^{-d_2}}_{\text{in money}} \left(K - S_0 e^{(r - \sigma^2/2)T + \sigma\sqrt{T}\cdot x}\right) \cdot \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx + \underbrace{\int_{-d_2}^{\infty} 0\, dx}_{\text{out of money}}$$

$$= K \int_{-\infty}^{-d_2} \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx - e^{rT} \int_{-\infty}^{-d_2} \left(\frac{1}{\sqrt{2\pi}}\right) S_0\, e^{-\frac{1}{2}(\sigma^2 T - 2\sigma\sqrt{T}\cdot x + x^2)}\, dx$$

$$= K \int_{-\infty}^{-d_2} \left(\frac{1}{\sqrt{2\pi}}\right)\left(e^{-x^2/2}\right) dx - e^{rT} \int_{-\infty}^{-d_2} \left(\frac{1}{\sqrt{2\pi}}\right) S_0\, e^{-\frac{1}{2}(x - \sigma\sqrt{T})^2}\, dx$$

$$\left(\text{Recall: } \Phi(x) = \int_{-\infty}^{x} \left(\frac{1}{2\pi}\right) e^{-x^2/2}\, dx\right)$$

let $u = x - \sigma\sqrt{T}$

$dx = du$

$$= K\,\Phi(-d_2) - e^{rT}(S_0)\int_{-\infty}^{-d_2-\sigma\sqrt{T}} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}(u)^2}\, du$$

$$= K\,\Phi(-d_2) - e^{rT}(S_0)\,\Phi(-d_1)$$

$$\Rightarrow V_0 = e^{-rT}\,\mathbb{E}\left((K - S_T)^+\right)$$

$$= e^{-rT} K\,\Phi(-d_2) - S_0\, e^{rT}\,\Phi(-d_1)$$