Sum of independent Gaussian random variables

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Exercise 2.1.2. Check that if X_1, X_2 are independent and standard Gaussian random variables, then $(X_1, X_2)^T$ is a Gaussian vector. Show that the random variable $a_1X_1 + a_2X_2$ is a Gaussian random variable with mean 0 and variance $a_1^2 + a_2^2$. Generalize your result for N independent and standard Gaussian random variables.

Proof. Since X_1, X_2 are independent and standard Gaussian random variables, each X_i has a probability density function given by $\Pi_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $i = 1, 2$.

In general, when X_1, X_2 are independent, the probability density function of $Y := a_1X_1 + a_2X_2$ is given in a convolutional form, namely

$$\Pi_Y(y) = \int_{\mathbb{R}} \Pi_{a_1X_1}(y-x)\,\Pi_{a_2X_2}(x)\,dx = \int_{\mathbb{R}} \frac{1}{|a_1a_2|}\Pi_{X_1}\left(\frac{y-x}{a_1}\right)\Pi_{X_2}\left(\frac{x}{a_2}\right)dx. \quad (1)$$

The integrand is calculated as:

$$\begin{aligned}
\frac{1}{|a_1a_2|}\Pi_{X_1}\left(\frac{y-x}{a_1}\right)\Pi_{X_2}\left(\frac{x}{a_2}\right) &= \frac{1}{2\pi|a_1a_2|}e^{-\frac{1}{2}\left(\frac{(y-x)^2}{a_1^2}+\frac{x^2}{a_2^2}\right)} \\
&= \frac{1}{2\pi|a_1a_2|}e^{-\frac{1}{2}\left(\frac{y^2}{a_1^2}-\frac{2yx}{a_1^2}+\frac{a_1^2+a_2^2}{a_1^2a_2^2}x^2\right)} \\
&= \frac{1}{2\pi|a_1a_2|}e^{-\frac{a_1^2+a_2^2}{2a_1^2a_2^2}\left(\left(x-\frac{a_2^2}{a_1^2+a_2^2}y\right)^2+\frac{a_1^2a_2^2}{(a_1^2+a_2^2)^2}y^2\right)} \\
&= \frac{1}{2\pi|a_1a_2|}e^{-\frac{a_1^2+a_2^2}{2a_1^2a_2^2}\left(x-\frac{a_2^2}{a_1^2+a_2^2}y\right)^2}e^{-\frac{y^2}{2(a_1^2+a_2^2)}}.
\end{aligned}$$

Therefore, the integral (1) can be written as

$$\begin{aligned}
\Pi_Y(y) &= \int_{\mathbb{R}} \frac{1}{|a_1a_2|}\Pi_{X_1}\left(\frac{y-x}{a_1}\right)\Pi_{X_2}\left(\frac{x}{a_2}\right)dx \\
&= \frac{1}{2\pi|a_1a_2|}e^{-\frac{y^2}{2(a_1^2+a_2^2)}}\int_{\mathbb{R}} e^{-\frac{a_1^2+a_2^2}{2a_1^2a_2^2}\left(x-\frac{a_2^2}{a_1^2+a_2^2}y\right)^2}dx \\
&= \frac{1}{2\pi|a_1a_2|}e^{-\frac{y^2}{2(a_1^2+a_2^2)}}\sqrt{\pi\frac{a_1^2a_2^2}{2(a_1^2+a_2^2)}} \\
&= \frac{1}{\sqrt{2\pi(a_1^2+a_2^2)}}e^{-\frac{y^2}{2(a_1^2+a_2^2)}},
\end{aligned}$$

where in the third equality we used Gaussian integral formula, $\int_{\mathbb{R}} e^{-a(x-b)^2}\,dx = \sqrt{\pi/a}$. So $\Pi_Y(y)$ is in the form of the probability density function of Gaussian random variable with mean 0 and variance $a_1^2 + a_2^2$, namely $N(0, a_1^2 + a_2^2)$.

The generalization of this result for N independent and standard Gaussian random variables, namely the claim that if $\{X_i\}_{i=1}^N$ are independent and standard Gaussian random variables then $\sum_{i=1}^N a_i X_i$ is a Gaussian random variable with mean 0 and variance $\sum_{i=1}^N a_i^2$, can be realized by induction. Let $Y_n = \sum_{i=1}^n a_i X_i$ and suppose Y_{N-1} is a Gaussian random variable with mean 0 and variance $\sum_{i=1}^{N-1} a_i^2$. Since

$$Y_{N-1} = N\left(0,\ \sum_{i=1}^{N-1} a_i^2\right) \quad \left(\Pi_{Y_{N-1}} = \frac{1}{\sqrt{2\pi \sum_{i=1}^{N-1} a_i^2}}\, e^{-\frac{y^2}{2\sum_{i=1}^{N-1} a_i^2}}\right),$$

Y_{N-1} divided by $(\sum_{i=1}^{N-1} a_i^2)^{\frac{1}{2}}$ is a standard Gaussian random variable N(0, 1). By the result we proved above, if Z_1, Z_2 are independent and standard Gaussian random variables, then $b_1 Z_1 + b_2 Z_2 = N(0,\ b_1^2 + b_2^2)$ for any b_1, b_2. Given that Y_N can be written as

$$Y_N = Y_{N-1} + a_N X_N,$$

by substituting b_1, Z_1, b_2, Z_2 with $(\sum_{i=1}^{N-1} a_i^2)^{\frac{1}{2}}$, $(\sum_{i=1}^{N-1} a_i^2)^{-\frac{1}{2}} Y_{N-1}$, a_N, X_N respectively, we get

$$Y_N = N\left(0,\ \sum_{i=1}^{N} a_i^2\right) \quad \left(\Pi_{Y_N} = \frac{1}{\sqrt{2\pi \sum_{i=1}^{N} a_i^2}}\, e^{-\frac{y^2}{2\sum_{i=1}^{N} a_i^2}}\right),$$

as desired.

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