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3.1 Brownian moments ([A] p.64)

Use the definition of Brownian motion to compute the following moments.

(a) $\mathbb{E}[B_t^6] = \frac{d^6}{d\theta^6}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0}$ by definition $\mathbb{E}[B_t^n] = \frac{d^n}{d\theta^n}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0}$

$= \frac{d^6}{d\theta^6}\left(e^{\frac{1}{2}\theta^2 t}\right)\Big|_{\theta=0}$ $\quad \because \mathbb{E}[e^{\theta B_t}] = e^{E[\theta B_t]+\frac{1}{2}V[\theta B_t]} = e^{0+\frac{1}{2}\theta^2 t} = e^{\frac{1}{2}\theta^2 t}$

$= \frac{d^5}{d\theta^5}\left(e^{\frac{1}{2}\theta^2 t}\theta t\right)\Big|_{\theta=0}$

$= \frac{d^4}{d\theta^4}\left(e^{\frac{1}{2}\theta^2 t}[t+\theta^2 t^2]\right)\Big|_{\theta=0}$

$= \frac{d^3}{d\theta^3}\left(e^{\frac{1}{2}\theta^2 t}[3\theta t^2+\theta^3 t^3]\right)\Big|_{\theta=0}$

$= \frac{d^2}{d\theta^2}\left(e^{\frac{1}{2}\theta^2 t}[3t^2+6\theta^2 t^3+\theta^4 t^4]\right)\Big|_{\theta=0}$

$= \frac{d}{d\theta}\left(e^{\frac{1}{2}\theta^2 t}[15\theta t^3+10\theta^3 t^4+\theta^5 t^5]\right)\Big|_{\theta=0}$

$= e^{\frac{1}{2}\theta^2 t}[15t^3+45\theta^2 t^4+15\theta^4 t^5+\theta^6 t^6]\Big|_{\theta=0}$

$= 15t^3$

Observe that

$\mathbb{E}[B_t^1] = e^{\frac{1}{2}\theta^2 t}\theta t\Big|_{\theta=0} = 0$

$\mathbb{E}[B_t^3] = e^{\frac{1}{2}\theta^2 t}[3\theta t^2+\theta^3 t^3]\Big|_{\theta=0} = 0$

$\mathbb{E}[B_t^5] = e^{\frac{1}{2}\theta^2 t}[15\theta t^3+10\theta^3 t^4+\theta^5 t^5]\Big|_{\theta=0} = 0$

By induction, one can show that

$\mathbb{E}[B_t^n] = 0$ if n = odd

Since all terms have θ while even n has a θ-independent term

(b) $\mathbb{E}[(B_{t_2}-B_{t_1})(B_{t_3}-B_{t_2})]$ if $t_1 < t_2 < t_3$

$= \mathbb{E}[B_{t_2}B_{t_3}] - \mathbb{E}[B_{t_1}B_{t_3}] - \mathbb{E}[B_{t_2}B_{t_2}] + \mathbb{E}[B_{t_1}B_{t_2}]$ $\quad$ since $\mathbb{E}[B_t B_s] = t \wedge s$

$= t_2 - t_1 - t_2 + t_1$

$= 0$

(c) $\mathbb{E}[B_s^2 B_t^2]$ if $s < t$

$$\left(\mathbb{E}[B_t^4] = \frac{d^4}{d\theta^4}\left(e^{\frac{1}{2}\theta^2 t}\right)\Big|_{\theta=0} = e^{\frac{1}{2}\theta^2 t}(3t^2+6\theta^2 t^3+\theta^4 t^4)\Big|_{\theta=0} = 3t^2\right)$$

$= \mathbb{E}[B_s^2(B_t-B_s+B_s)^2]$

$= \mathbb{E}[B_s^2(B_t-B_s)^2] + \mathbb{E}[2B_s^3(B_t-B_s)] + \mathbb{E}[B_s^4]$

$= \mathbb{E}[B_s^2]\,\mathbb{E}[(B_t-B_s)^2] + 2\underbrace{\mathbb{E}[B_s^3]}_{=0}\underbrace{\mathbb{E}[B_t-B_s]}_{=0} + \underbrace{\mathbb{E}[B_s^4]}_{=3s^2}$ $\quad \because B_s$ and $B_t - B_s$ are independent and $\mathbb{E}(B_t^n) = 0$ where n is odd

$= \mathbb{E}[B_s B_s]\,\mathbb{E}[B_t B_t - 2B_t B_s + B_s B_s] + 3s^2$ $\quad \because \mathbb{E}[B_t B_s] = t \wedge s$

$= s(t-2s+s) + 3s^2$

$= 2s^2 + st$

(d) $\mathbb{E}[B_s B_t^3]$ if $s < t$

$$\mathbb{E}[B_t^2] = \frac{d^2}{d\theta^2}\left(e^{\frac{1}{2}\theta^2 t}\right)\Big|_{\theta=0} = e^{\frac{1}{2}\theta^2 t}(t+\theta^2 t^2)\Big|_{\theta=0} = t$$

$= \mathbb{E}[B_s(B_t-B_s+B_s)^3]$

$= \mathbb{E}[B_s(B_t-B_s)^3] + 3\mathbb{E}[B_s^2(B_t-B_s)^2] + 3\mathbb{E}[B_s^3(B_t-B_s)] + \mathbb{E}[B_s^4]$

$= \underbrace{\mathbb{E}[B_s]}_{=0}\underbrace{\mathbb{E}[(B_t-B_s)^3]}_{=0} + 3\underbrace{\mathbb{E}[B_s^2]}_{=s}\mathbb{E}[(B_t-B_s)^2] + 3\underbrace{\mathbb{E}[B_s^3]}_{=0}\underbrace{\mathbb{E}[B_t-B_s]}_{=0} + \mathbb{E}[B_s^4]$ $\quad \because B_s$ and $B_t - B_s$ are independent and $\mathbb{E}(B_t^n) = 0$ where n is odd

$= 3s(t-s) + 3s^2 = 3st$

(e) $\mathbb{E}\left[B_s^{100} B_t^{101}\right] = \mathbb{E}\left[B_s^{100}(B_t - B_s + B_s)^{101}\right]$ use binomial formula

$= \sum_{k=0}^{101} \binom{101}{k} \mathbb{E}\left[B_s^{100}(B_t - B_s)^{101-k} B_s^{k}\right] \leftarrow (B_t - B_s + B_s)^{101} = \sum_{k=0}^{101} \binom{101}{k} (B_t - B_s)^{101-k} B_s^{k}$

$= \mathbb{E}\left[B_s^{100}(B_t - B_s)^{101}\right] + 101\,\mathbb{E}\left[B_s^{101}(B_t - B_s)^{100}\right] + \dots$

$+ 101\,\mathbb{E}\left[B_s^{200}(B_t - B_s)\right] + \mathbb{E}\left[B_s^{201}\right]$

$\because$ B_s and $B_t - B_s$ are independent

$= \mathbb{E}\left[B_s^{100}\right] \underbrace{\mathbb{E}\left[(B_t - B_s)^{101}\right]}_{=0} + 101\,\mathbb{E}\underbrace{\left[B_s^{101}\right]}_{=0} \mathbb{E}\left[(B_t - B_s)^{100}\right] + \dots$

$+ 101\,\mathbb{E}\left[B_s^{200}\right] \underbrace{\mathbb{E}\left[B_t - B_s\right]}_{=0} + \underbrace{\mathbb{E}\left[B_s^{201}\right]}_{=0}$

All terms are combination of odd and even

and $\mathbb{E}(B_t^{n}) = 0$ if $n = \text{odd}$, and similarly $\mathbb{E}\left((B_t - B_s)^{n}\right) = 0$ if $n = \text{odd}$

$\Rightarrow$ all terms are 0

$\therefore\ \mathbb{E}\left[B_s^{100} B_t^{101}\right] = 0$

In fact, one can argue that for all $n \in \mathbb{N}$

$\mathbb{E}\left[B_s^{n} B_t^{n+1}\right] = 0$ as $n + n + 1 = 2n + 1$ is odd